(Q-1) The number 3 can be expressed as a sum of one or more positive integers, taking order into account, in four ways, namely, 3, 1 + 2, 2 + 1, and 1 + 1 + 1. Show that any positive integer \( n \) can be so expressed in \( 2^{n-1} \) ways.

(Q-2) In how many ways can 10 be expressed as a sum of 5 nonnegative integers, when order is taken into account? (Hint: Find an equivalent problem in which the phrase “5 nonnegative integers” is replaced by “5 positive integers”.)

(Q-3) Given \( n \) objects arranged in a row. A subset of these objects is called unfriendly if no two of its elements are consecutive. Show that the number of unfriendly subsets with \( k \) elements is \( \binom{n-k+1}{k} \).

(Hint: Adopt an idea similar to that used in Larson 1.3.6.)

(Q-4) Let \( a_1, a_2, \ldots, a_n \) be a permutation of the set \( S_n = \{1, 2, \ldots, n\} \). An element \( i \) of \( S_n \) is called a fixed point of this permutation if \( a_i = i \). A derangement of \( S_n \) is a permutation of \( S_n \) having no fixed points. Let \( g_n \) be the number of derangements of \( S_n \). Show that

\[
g_1 = 0, \quad g_2 = 1, \quad g_n = (n-1)(g_{n-1} + g_{n-2}), \quad \text{for } n > 2.
\]

(Hint: a derangement either interchanges the first element with another element, or it doesn’t.)

(Q-5) Sum each of the following:

(a) \( 1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} \).

(b) \( 1 \times 2 \binom{n}{2} + 2 \times 3 \binom{n}{3} + \cdots + (n-1)n \binom{n}{n} \).

(c) \( \binom{n}{1} + 2^2 \binom{n}{2} + 3^2 \binom{n}{3} + \cdots + n^2 \binom{n}{n} \).

(d) \( \binom{n}{1} - 2^2 \binom{n}{2} + 3^2 \binom{n}{3} - \cdots + (-1)^{n+1} n^2 \binom{n}{n} \).

(Q-6) Show that

(a) \( \sum_{r=0}^{s} \binom{r}{n} \binom{s}{n+1} + \binom{r}{2} \binom{s}{n+2} + \cdots + \binom{r}{n+r} \binom{s}{n+r} = \binom{r+s}{s-n} \).

(b) \( \sum_{r=0}^{s} \binom{n}{r}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n} \).