(Q-1) Let $p > 5$ be a prime. Show that the number $\overbrace{11\ldots11}^{p-1}$ (with $p-1$ digits of 1) is divisible by p.

(Q-2) Calculate $2^{998} \pmod{121}$.

(Q-3) Let $p > 5$ be a prime. Prove that $p^8 \equiv 1 \pmod{240}$.

(Q-4) Find polynomials $F(x)$ and $G(x)$ such that

$$(x^8 - 1)F(x) + (x^5 - 1)G(x) = x - 1.$$

(Q-5) Show that $x^{4a} + x^{4b+1} + x^{4c+2} + x^{4d+3}$, a, b, c, d positive integers, is divisible by $x^3 + x^2 + x + 1$. (Hint: $x^3 + x^2 + x + 1 = (x^2 + 1)(x + 1)$.)

(Q-6) Factor $x^8 + x^4 + 1$ into irreducibles

(a) over the rationals,

(b) over the reals,

(c) over the complex numbers.

(Q-7) Here are two results that are useful in factoring polynomials with integer coefficients into irreducibles.

Rational-Root Theorem. *If $P(x) = a_n x^n + \cdots + a_0$ is a polynomial with integer coefficients, and if the rational number r/s (r and s are relatively prime) is a root of $P(x) = 0$, then r divides a_0 and s divides a_n.*

Gauss' Lemma *Let $P(x)$ be a polynomial with integer coefficients. If $P(x)$ can be factored into a product of two polynomials with rational coefficients, then $P(x)$ can be factored into a product of two polynomials with integer coefficients*

(a) Let $f(x) = a_n x^n + \cdots + a_0$ be a polynomial of degree n with integral coefficients. If a_0, a_n and $f(1)$ are odd, prove that $f(x) = 0$ has no rational roots.

(b) For what integer a does $x^2 - x + a$ divide $x^{13} + x + 90$?

(Q-8) (a) Let $F(x)$ be a polynomial over the real numbers. Prove that a is a zero of multiplicity $m + 1$ if and only if $F(a) = F'(a) = \cdots = F^{(m)}(a) = 0$ and $F^{(m+1)}(a) \neq 0$.

(b) The equation $f(x) = x^n - nx + n - 1 = 0$, $n > 1$, is satisfied by $x = 1$. What is the multiplicity of this root?

(Q-9) Given r, s, t are the roots of $x^3 + ax^2 + bx + c = 0$,

(a) Evaluate $1/r^2 + 1/s^2 + 1/t^2$, provided $c \neq 0$.

(b) Find a polynomial equation whose roots are r^2, s^2, t^2.

(Q-10) Prove that if p is a prime, then $ab^p - ba^p$ is divisible by p.