(Q-1) (a) Use induction to prove that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$

(b) Use induction to prove that $2!4!\ldots(2n)! \geq ((n+1)!)^n$.

(Q-2) A group of n people play a round-robin tournament. Each game ends in either a win or a loss. Show that it is possible to label the players $P_1, P_2, \ldots, P_n$ in such a way that P_1 defeated P_2, P_2 defeated P_3,...,P_{n-1} defeated P_n.

(Q-3) Show that if a round-robin tournament has an odd number of teams, it is possible for every team to win exactly half its games. (Hint: Assuming the result for $2n-1$ teams, show how the same thing could happen when 2 teams are added.)

(Q-4) For all $0 \leq x \leq \pi$ and all non-negative integers n, prove that

$$|\sin nx| \leq n \sin x.$$

(Q-5) Show that every positive integer can be written as a sum of distinct Fibonacci numbers.

(Q-6) Let A be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \ldots, 100$. Prove that there must be two distinct integers in A whose sum is 104.

(Q-7) (a) Let S be a square region (in the plane) of side length 2 inches. Show that among any nine points in S, there are three which are the vertices of a triangle of area $\leq \frac{1}{2}$ square inch.

(b) Nineteen darts are thrown onto a dartboard which has the shape of a regular hexagon with side length one foot. Show that two darts are within $\frac{\sqrt{3}}{3}$ feet of one another.

(Q-8) Fifteen chairs are evenly placed around a circular table on which are name cards for fifteen guests. The guests fail to notice these cards until after they have sat down, and it turns out that no one is sitting in front of their own card. Prove that the table can be rotated so that at least two of the guests are simultaneously correctly seated.

(Q-9) Let X be any real number. Prove that among the numbers

$$X, 2X, \ldots, (n-1)X$$

there is at least one that differs from an integer by at most $\frac{1}{n}$.

(Q-10) Prove that in a group of six people there are either three mutual friends or three mutual strangers. (Hint: Represent the people by the vertices of a regular hexagon. Connect vertices by a red line segment if the couple represented by these vertices are friends; otherwise, connect them with a blue line segement. Consider one of the vertices, say A. At least three line segments emanating from A have the same color. There are two cases to consider.)