Induction.

(1) $\frac{1}{n+1}+\cdots+\frac{1}{2n}=1-\frac{1}{2}+\frac{1}{3}-\cdots+\frac{1}{2n-1}-\frac{1}{2n}$.
(2) $2^n \geq n^2 \qquad \forall n \geq 4$.
(3) Complement of n lines in $\mathbb{R}^2$ can be checkerboard colored.
(4) Assume Bertrand Postulate, there is a prime p between x and $2x$ for all integers x. Prove that every natural number can be written as a sum of distinct primes or 1.
(5) Fibonacci sequence $F_1=F_2=1$, $F_{n+1}=F_n+F_{n-1}$. Show $F_{2n+1}=F_{n+1}^2+F_n^2$.
(6) Every triangle can be cut into $n \geq 6$ triangles similar to it.
(7) $V-E+F=2$ for connected planar graphs.

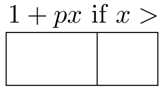
Pigeonhole Principle.

(1) 5 points chosen in a triangle of side length 2; some two are within distance 1.
(2) 50 distinct positive integers chosen less than 99. Some two add up to 99.
(3) n people are at a party. Show some two know the same number of people.
(4) Each point in the plane is colored red-blue. Some two points at distance 1 have same color. Also works with 3 colors.
(5) 5 points are chosen from a 2D lattice Λ. The midpoint of some two is also on Λ.
(6) $\{x\} \in [0,1)$ is the fractional part. For irrational α, $\{n\alpha\}$ is dense in $[0,1]$.

Inequalities.

(1) AM-GM, proof by induction.
(2) Weighted AM-GM.
(3) Generalization. If f continuous with $f(\frac{x_1+x_2}{2}) \geq \frac{f(x_1)+f(x_2)}{2}$, then $f(\frac{\sum p_i x_i}{\sum p_i}) \geq \frac{\sum p_i f(x_i)}{\sum p_i}$.
(4) These sort of functions are called concave. Appear from C^2 functions with $f'' \leq 0$. Jensen's inequality. Proof uses mean value theorem.

$$(f(x_2)-f(\frac{x_1+x_2}{2}))-(f(\frac{x_1+x_2}{2})-f(x_1))=\frac{x_1-x_2}{2}(f'(y_2)-f'(y_1))=\frac{x_1-x_2}{2}(y_2-y_1)f''(z)\leq 0.$$

(5) Apply to $f(x)=\ln x$ to get AM-GM by applying exp to both sides.
(6) Basic inequality, if f is C^1 and $f' \geq 0$, then f is increasing.
(7) Prove $2\sin x+\tan x \geq 3x$ for $0 \leq x < \pi/2$.
(8) Taylor expansion. $(1+x)^p \leq 1+px$ if $x>-1, 0<p<1$.
(9) Maximize area ab for fencing  with total fence length $3a+2b=1000$.
(10) Generalizations of AM-GM. $\left(\frac{\sum p_i x_i^k}{\sum p_i}\right)^{1/k}$ is increasing in k. Mention special cases $k=0,\pm 1,2,\pm\infty$ (GM, AM, HM, QM, max, min).
(11) For $a,b,c>0$,

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{3}{2}.$$

(12) Cauchy-Schwarz inequality. Difference is a sum of squares $\sum_{i<j}(a_ib_j-a_jb_i)^2$. Equality if proportional.
(13) Geometric interpretation. $|\vec{a}\cdot\vec{b}|=|\vec{a}||\vec{b}|\cos\theta$.
(14) Cauchy-Schwarz is equivalent to weighted AM-QM.
(15) Triangle inequality. For a,b,c sides of a triangle, show $(a+b-c)(b+c-a)(c+a-b)\leq abc$. Useful substitution $x=b+c-a$, etc.
(16) Sides of a triangle. Show $2(a^2+b^2+c^2)\leq(a+b+c)^2$.
(17) Holder inequality (generalization of Cauchy-Schwarz). For $1/p+1/q=1$,

$$\sum|a_ib_i|\leq(\sum_i|a_i|^p)^{1/p}(\sum_i|b_i|^q)^{1/q}$$

(18) Minkowski inequality (generalization of triangle inequality). For $p\leq 1$,

$$(\sum|a_i+b_i|^p)^{1/p}\leq(\sum|a_i|^p)^{1/p}+(\sum|b_i|^p)^{1/p}.$$

Number theory.

(1) Euclidean algorithm for gcd. Unique decomposition $a=qb+r$ (Euclidean domain). Corollary: $\{sa+tb \mid s,t\in\mathbb{Z}\}=\mathbb{Z}\langle\gcd(a,b)\rangle$. Do for $a=90,b=65$.

(2) Find all solutions of $ax + by = c$ ($x = x_0 + bt/d, y = y_0 - at/d$). Do $18x + 32y = 6$.
(3) Prove $\frac{21n+4}{14n+3}$ is irreducible.
(4) Prove $\sqrt{3}$ irrational. Do $\sqrt{2} + \sqrt{3}$.
(5) Unique factorization into primes. Need $p \mid ab \implies p \mid a$ or $p \mid b$.
(6) Prove $\gcd(a, b)\text{lcm}(a, b) = ab$.
(7) Modular arithmetic, residue classes $\pmod{n}$, $\mathbb{Z}/n$. Notation, $x \equiv y \pmod{n}$. Calculate $142 + (15 \cdot 72) \pmod{7}$.
(8) Every odd square is $1 \pmod{8}$.
(9) Infinitely many primes. Euclid $p_1 \cdots p_k + 1$. Infinitely many primes $3 \pmod{4}$. $4p_1 \cdots p_k - 1$.
(10) Last digit of $7^{\text{current year } 20xx}$.
(11) Divisibility rule of 9 and 11. Calculate 1745002145 modulo 9 and 11.
(12) Fermat's little theorem. $a^p \equiv a \pmod{p}$, better version: $a^{p-1} \equiv 1 \pmod{p}$ for $\gcd(a, p) = 1$. Calculate $2^{233} \pmod{47}$.
(13) Generalization $a^{\varphi(n)} \equiv 1 \pmod{n}$, φ is Euler's totient function. Proof: If $b_1, \ldots, b_{\phi(n)}$ are coprime residue classes $\pmod{n}$, so are $ab_1, \ldots, ab_{\phi(n)}$.
(14) $n = p_1^{a_1} \cdots p_k^{a_k}$. Then $\varphi(n) = n(1 - 1/p_1) \cdots (1 - 1/p_k)$ by principle of inclusion exclusion. Calculate $17^{81} \pmod{60}$.
(15) Number of divisors $\tau(n) = (1 + a_1) \cdots (1 + a_k)$. Find smallest integer with exactly 28 divisors.
(16) Sum of divisors $\sigma(n) = \prod_i \frac{p_i^{a_i} - 1}{p_i - 1}$. Do $n = 1000$.
(17) Highest power of 3 dividing $100!$, $\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \cdots$.
(18) For any 55 numbers from $\{1, \ldots, 100\}$, some two must differ by 9.
(19) Chinese remainder theorem. $m_1, \ldots, m_n$ pairwise relatively prime, and $a_i \in \mathbb{Z}$. There is some x with $x \equiv a_i \pmod{m_i}$.
(20) There are a million consecutive integers none of which is a prime power.
(21) For p prime, each prime factor of $2^p - 1$ is bigger than p (so infinitely many primes).
(22) If m odd, $m \mid 2^{(m-1)!} - 1$. $\varphi(m) \mid (m - 1)!$.

Algebra.

(1) Groups. Typical example symmetric group S_n. Also $(\mathbb{R}, +), (\mathbb{R}^*, \cdot)$.
(2) If $(ab)^3 = 1$, show $(ba)^3 = 1$.
(3) If $aba = ba^2b, a^3 = 1, b^3 = 1$, show $b = 1$.
(4) Ring, always with 1. Typical example $n \times n$ matrices $M_n(\mathbb{R})$. Define center $Z(R)$.
(5) Commutative ring. Typical examples $\mathbb{Z}, \mathbb{Z}/n$.
(6) If $x^2 - x \in Z(R)$ for all x, then R is commutative.
(7) Integral domain if no zero divisor. Typical example $\mathbb{Z}, \mathbb{Z}[\sqrt{2}], \mathbb{Z}[X]$, not $\mathbb{Z}/6$.
(8) Field. Example $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}/p, \mathbb{Q}[\sqrt{2}]$.
(9) $D[X]$ is a domain if D is.
(10) Euclidean domain, $\mathbb{F}[X]$ but not $\mathbb{Z}[X]$ (division algorithm in $\mathbb{Z}[X]$ works for monic polynomials). Example: $x^5 + 3x^2 + 1 = (2x^3 - x + 1)(x^2/2 + 1/4) + (5x^2/2 + x/4 + 3/4)$.
(11) Euclidean domains are unique factorization domains, as is $D[X]$ for any integral domain D upto multiplication with elements in D (can work in quotient field of D).
(12) Find $\gcd(x^8 - 1, x^5 - 1) = x - 1$ by Euclidean algorithm.
(13) Find $P \in \mathbb{Q}[X]$ with $x^2 + 1 \mid P(x)$ and $x^3 + x^2 + 1 \mid P(x) + 1$.
(14) Over an integral domain, factor theorem: $P(\alpha) = 0$ iff $x - \alpha \mid P(x)$.
(15) Unique factorization for $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$. Fundamental theorem of algebra for $\mathbb{R}, \mathbb{C}$.
(16) For what n is $x^{4n} - 2x^n + 1$ divisible by $x^2 + 1$?
(17) Identity theorem. If $\deg \le n$ polynomials P, Q have $P(\alpha_i) = Q(\alpha_i)$, $i = 0, \ldots, n$, then $P = Q$.
(18) Find all $P \in \mathbb{R}[X]$ with $P(x^2 + 1) = P(x)^2 + 1$ and $P(0) = 0$.
(19) Gauss' Lemma for $\mathbb{Z}$ (a rational factorization produces an integer factorization)
(20) Viete relations, relating symmetric polynomials of roots to coefficients. Find polynomial for the squares of roots of $x^2 + ax + b$.
(21) $a_n x^n + \cdots + a_3 x^3 + x^2 + x + 1$ does not have all real roots.
(22) Wilson's theorem $(p - 1)! \equiv -1 \pmod{p}$.

Series.

(1) $a + ax + \cdots + ax^n$, then as $x \to \infty$. Define partial sums.
(2) $\cos\theta + \cdots + \cos n\theta$.
(3) Telescoping. $\sum a_n$, with $a_n = b_{n-1} - b_n$. If $b_n \to 0$, then converges.
(4) $\sum \frac{1}{n(n+1)}$.
(5) Prove $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$.
(6) $\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1}$.
(7) Series for $(1+x)^n$. $\sum_i \binom{n}{i}$, $\sum_i (-1)^i \binom{n}{i}$, $\sum_k \binom{m}{r-k}\binom{n}{k} = \binom{m+n}{r}$.
(8) Taylor series. Recalls (finite) Taylor's theorem. Do $e^x, \cos x, \sin x, \log(1+x), (1+x)^r$ (for arbitrary $r \neq 0$).
(9) Geometric again $\sum x^n = 1/(1-x)$. Do $\sum nx^{n-1}$, $\sum x^n/n$.
(10) e is irrational.
(11) Power series $\sum_n a_n x^n$. Radius of convergence.
(12) $\sum F_n/3^n$. Then write $f(x) = \sum_{n\geq 0} F_n x^n = x + x^2 + 2x^3 + \cdots$, and by recurrence, $f(x) = \frac{x}{1-x-x^2}$.

Combinatorics.

(1) $n!$, permutation of n elements, number of bijective functions $\{1, \ldots, n\} \to \{1, \ldots, n\}$.
(2) n^k, k ordered (not necessarily distinct) elements from $\{1, \ldots, n\}$, number of functions $\{1, \ldots, k\} \to \{1, \ldots, n\}$. Number of 4-letter words 26^4.
(3) $n!/(n-k)!$, k ordered distinct elements from $\{1, \ldots, n\}$, number of injective functions $\{1, \ldots, k\} \to \{1, \ldots, n\}$.
(4) $\binom{n}{k}$, k unordered distinct elements from $\{1, \ldots, n\}$, number of (strictly) increasing functions $\{1, \ldots, k\} \to \{1, \ldots, n\}$.
(5) $\binom{n+k-1}{k}$, k unordered (not necessarily distinct) elements from $\{1, \ldots, n\}$, number of (not necessarily strictly) increasing functions $\{1, \ldots, k\} \to \{1, \ldots, n\}$, same as number of strictly increasing functions $\{1, \ldots, k\} \to \{1, \ldots, n+k-1\}$.
(6) $\binom{n}{k} = \binom{n}{n-k}$, chosen set vs its complement.
(7) $k\binom{n}{k} = n\binom{n-1}{k-1}$, a set and an element in it.
(8) $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, a fixed element is chosen or not, Pascal's triangle.
(9) $\sum_k \binom{m}{r-k}\binom{n}{k} = \binom{m+n}{r}$, choose r elements from m elements and n elements.
(10) $\sum k\binom{n}{k}$, differentiate $(1+x)^n$, or use $k\binom{n}{k} = n\binom{n-1}{k-1}$.
(11) $\sum \frac{1}{k+1}\binom{n}{k}$, integrate, or use $\frac{1}{n+1}\binom{n+1}{k+1} = \frac{1}{k+1}\binom{n}{k}$.
(12) $\binom{n+k+1}{k} = \binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+k}{k}$. Induct on k. Alternatively, choose a subset S of size k from $\{1, \ldots, n+k+1\}$, and break it up by the largest number outside of S.
(13) Principle of inclusion exclusion.
(14) Number of surjective functions $\{1, \ldots, k\} \to \{1, \ldots, n\}$. If A_i is the set of functions that don't hit i, then the number is $n^k - |A_1 \cup \cdots A_n|$. Use inclusion exclusion to get $\sum_i (-1)^i \binom{n}{i}(n-i)^k$.

Recurrences.

(1) Fibonacci sequence $f(x) = \sum F_n x^n$, $f(x) = \frac{x}{1-x-x^2}$. Break into partial fractions, $\frac{x}{1-x-x^2} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$, where α, β solutions to $z^2 - z - 1$ (characteristic equation). Get $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1+\sqrt{5}}{2}\right)^n\right)$.
(2) Number of ways of tiling $2 \times n$ board with 2×1 dominoes, F_{n+1}.
(3) General linear recurrences. Distinct roots, repeated roots.
(4) $a_n = 4a_{n-1} - 4a_{n-2}$, $a_0 = a_1 = 1$.
(5) $a_n = 3a_{n-1} - 4a_{n-3}$, $a_0 = 30, a_1 = -10, a_2 = 20$.
(6) Linear recurrence with additional term. General solution plus one particular solution. $a_n = 3a_{n-1} + 1$.
(7) Putnam problem, $a_1 = 1, a_2 = 2, a_3 = 24$, $a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}$. Show a_n is an integer divisible by n.
(8) Catalan numbers, C_n is number of ways of parenthesizing $n+1$ variables with $n-1$ bracket pairs, number of binary trees with $n+1$ leaves, number of crossingless matchings on $2n$ strands, number of paths from $(0,0)$ to (n,n) on or below the diagonal.
(9) Recurrence $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k-1}$. So if $f(x) = \sum C_n x^n$, $1 + xf(x)^2 = f(x)$, $f(x) = \frac{1-\sqrt{1-4x}}{2x}$. Do expansion to get $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Probability.

(1) Probability a function $\{1,2,3\} \to \{1,2,3,4,5\}$ is injective.
(2) Cashier (with no money initially) is selling $2n$ \$5 tickets, and n people bring \$5 notes, and n people bring \$10 notes. Probability of success $\frac{1}{n+1}$.
(3) Probability 2 people have the same birthday, more likely than not at 23 people.
(4) Real number picked from $[0,1]$, probability it is less than $1/3$? Need to assume picked uniformly.
(5) Probability a point from $[0,1]^2$ is within distance $1/5$ to the middle?
(6) Pick $x, y \in [0,1]$, probability $y \leq x^2$? Assume picked uniformly and independently.
(7) Two people arrive (uniformly) between 4 and 5, and each waits 15min and then leaves. Probability they will meet.
(8) Length one needle dropped on infinitely many horizontal lines at distance 2 apart. Probability it hits a line.
(9) Cut a length d bar into three pieces. Probability they form a triangle. If we picked 2 points x, y (wlog $x < y$) simultaneously and cut there, each of the sidelengths $x, y-x, d-y$ has to be less than the semiperimeter $d/2$. But if first cut at one point x (wlog $x > d/2$), then cut the larger piece at αx, then each of the side lengths $\alpha x, (1-\alpha)x, d-x$ is less than $d/2$.

Geometry.

(1) Medians of a triangle intersect by Euclidean geometry.
(2) Same using coordinate geometry. Take $A = (a_1, a_2)$, similarly.
(3) By translation, rotation, scaling, can assume $B = (0,0), C = (1,0)$.
(4) Drop perpendiculars PA, PB, PC from $P = (a,b)$ to a parabola. The centroid of ΔABC is on the axis.
(5) Find the slope one tangents to the ellipse $3x^2 + y^2 = 3$.
(6) Vector addition, points between two points, $\alpha\vec{v} + (1-\alpha)\vec{w}$.
(7) Medians intersect using vectors.
(8) D, E, F chosen on the sides of ΔABC with $AF/AB = BD/BC = CE/CA$. Centroids of ΔABC and ΔDEF agree.
(9) $AB = AC$, D midpoint of BC, DE perpendicular to AC, F midpoint of DE. Prove $AF \perp BE$.
(10) Suppose altitudes AP and BQ of a tetrahedron $ABCD$ are coplanar. Show $AB \perp CD$.
(11) $ABCD$ parallelogram, F midpoint of CD, AF intersects BD in E. Prove $DE = DB/3$.
(12) Complex conjugates, Cartesian coordinates for addition, polar coordinates for multiplication, unit circle, roots of unity.
(13) Find z_3, given two other points z_1, z_2 of an equilateral triangle, $z_3 = -z_1\omega - z_2\omega^2$.
(14) Draw equilaterals triangles BDC, CEA, AFB outside a triangle ABC. Show centroids of the three new triangles form an equilateral triangle.
(15) P, Q, R, S centers of squares drawn outside a quadrilateral $ABCD$. Show $PR \perp QS$ and $|PR| = |QS|$.
(16) Regular polygon $A_1A_2\cdots A_n$ of circumradius r, P some other point on the circumcircle. Show $\sum_k |PA_k|^2 = 2nr^2$.

Analysis.

(1) Continuity, different definitions, $\sin\frac{1}{x}$ discontinuous.
(2) Intermediate value property (IVT), extreme value property.
(3) Given two polygons, there is a line that cuts both in equal areas.
(4) Continuous f with $f(x+y) = f(x) + f(y)$ must be $f(x) = ax$.
(5) Continuous f with $f(x) = f(x^2)$ must be constant.
(6) A runner runs 6 miles ins 30 minutes. Show he ran one mile in exactly 5 minutes.
(7) Find all continuous functions $f\colon [0,\infty) \to \mathbb{R}$ with $f(xy) = xf(y) + yf(x)$.
(8) Example of (discontinuous) function with $f(x+y) = f(x) + f(y)$ but $f(x) \neq ax$.
(9) Derivatives. $x^2 \sin\frac{1}{x}$, f' exists but is not continuous.
(10) Rolle's theorem, mean value theorem, Taylor's theorem.
(11) $f = x^3 - 3x + a$ cannot have more than one zero in $[-1,1]$.
(12) Show $4ax^3 + 3bx^2 + 2cx - (a+b+c)$ has a root in $[0,1]$.
(13) P, Q points on unit circle, R perpendicular from Q to the tangent through P, maximize area of ΔPQR.
(14) Putnam problem, show all roots of $P(x) = 1 + 8x + 16x^2 + 8x^3 + x^4$ are real.
(15) Find number of solutions of $x^2 - x\sin x - \cos x = 0$.

(16) Find $\int_2^4 \frac{\sqrt{\ln(9-x)}\,dx}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}$.
(17) L'Hospital, f/g, both going to zero, or both going to infinity, $\lim_{x\to 0}(\frac{1}{\sin x} - \frac{1}{x})$.
(18) $\lim_{n\to\infty}(1+\frac{1}{n})^n$.
(19) $\lim_{x\to 0} x^x$,
(20) Fundamental theorem of calculus, $\lim_{x\to 0}\frac{1}{x}\int_0^x (1+\sin 2t)^{1/t}dt$.
(21) Find all f satisfying $f(x) = \int_0^x f(t)dt + 1$.
(22) $\int_0^x f(t)dt = \int_x^1 t^2 f(t)dt + x^{16}/8 + x^{18}/9 + C$.
(23) $\lim_{n\to\infty}\big(\frac{1}{2n+1} + \cdots + \frac{1}{3n}\big)$.
(24) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{k^2+n^2}}$.
(25) $\lim_{n\to\infty}\frac{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}}{\sqrt{n^3}}$.
(26) Find integral part (floor) of $S = \sum_1^{10^9} n^{-2/3}$.
(27) f differentiable, $f(0) = 0$, and f' strictly increasing (don't know if f'' exists). Show $f(x)/x$ is strictly increasing for $x > 0$.
(28) Find $I = \int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$.
(29) Putnam problem, $f\colon [1,3]\to\mathbb{R}$, $|f(x)|\le 1$, $\int_1^3 f(x)dx = 0$. Maximum value of $\int_1^3 \frac{f(x)}{x}dx$.