Functions

1. Define a function f from the set of natural numbers $\{1, 2, 3, \dots\}$ to the set of even natural numbers $\{2, 4, 6, \dots\}$ by letting $f(x) = 2x$. Show carefully that f is a bijection and conclude that these two sets have the same cardinality, even though one is a subset of the other.

(Remember that 'surjective' or 'onto' means that the function hits everything in the target space; 'injective' or 'one-to-one' means that everything that is hit is hit only once; and 'bijective' means that both of these happen).

Equivalence relations

For each of the following, determine whether or not it is an equivalence relation on the set $S = \{1, 2, \dots, 100\}$. If not, say why not. If yes, describe the equivalence classes.

(Recall that an 'equivalence relation' is a relation that obeys:

- Reflexivity: $x \sim x$ for every x.
- Symmetry: If $x \sim y$, then $y \sim x$.
- Transitivity: If $x \sim y$ and $y \sim z$, then $x \sim z$.

1. $a \sim b$ always.
2. $a \sim b$ if $a - b$ is a multiple of 7. (Define a 'multiple of 7' to be any number of the form $7k$, for k an integer. So the multiples of 7 are $\{\dots, -14, -7, 0, 7, 14, \dots\}$).
3. $a \sim b$ if $ab \geq a + b$.
4. $a \sim b$ if $|a - b| < 3$.
5. $a \sim b$ if $a + b$ is even.
6. Fix a function $f : S \to T$ for some set T; then $a \sim b$ if $f(a) = f(b)$.
7. Fix a function $f : S \to T$ for some set T, and an equivalence relation $\equiv$ on T. Then $a \sim b$ if $f(a) \equiv f(b)$.

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We break up the proof into two parts. First, we show that f is injective. That means we need to prove

$$\text{If } 2x = 2y, \text{ then } x = y.$$

But if $2x = 2y$, then we can just divide both sides by 2 to conclude $x = y$. So f is in fact injective.

Next we show that f is surjective. That means we need to prove

$$\text{For any even natural number } y, \text{ there is a natural number } x \text{ such that } y = 2x.$$

But there is such a natural number, namely $y/2$. So the function is also surjective.

Thus f is a bijection between the two sets, and if there is a bijection the sets must have the same cardinality.

Equivalence relations

1. Yes, this is an equivalence relation (the fact that any two elements are equivalent means that the conclusions of each of the axioms are satisfied). There is one equivalence class, and it is all of S.

2. Yes, this is an equivalence relation. For any x, $x - x = 0$ is a multiple of 7, so we have reflexivity. And if $x - y$ is a multiple of 7, so is $y - x$ (since the additive inverse of a multiple of 7 is still a multiple of 7). Lastly, if $x - y$ and $y - z$ are multiples of 7, so is their sum $x - y + y - z = x - z$. So this is an ER. The equivalence classes are the sets of things which have the same remainder upon division by 7, namely

$$\begin{aligned}&\{1, 8, 15, 22, \ldots\}\\ &\{2, 9, 16, 23, \ldots\}\\ &\ldots\\ &\{7, 14, 21, 28, \ldots\}\end{aligned}$$

   In fact, this equivalence relation is so well-behaved that we can define addition and multiplication on the equivalence classes, turning this into the integers modulo 7.

3. Not an ER as $1 \not\sim 1$. But it is symmetric and transitive!

4. Not an ER as $34 \sim 36$ and $36 \sim 37$ but $34 \not\sim 37$.

5. Yes, this is an ER. We check:

   - Symmetry: for any x, $x + x = 2x$ is even, so $x \sim x$.
   - Reflexivity: for any x, y, if $x + y$ is even, so is $y + x$; thus $x \sim y \implies y \sim x$.

- Transitivity: If $x+y$ and $y+z$ are both even, so is $(x+y)+(y+z) = x+z+2y$. But differences of even numbers are even, so $x+z$ is also even. Thus we get transitivity too.

If we notice that $a+b$ is even if and only if $a-b$ is even, we can see that this is qualitatively the same as (2). The equivalence classes are the sets of odd and even numbers.

6. Yes, this is an ER (It's a special case of (7) so I won't do it separately). Here the equivalence classes are the inverse images of the elements of T:

$$\{f^{-1}(\{t\}) : t \in T\}.$$

7. Yes, this is an ER. All the properties are pulled back by f:

- Symmetry: for any x, $f(x) \equiv f(x)$, so $x \sim x$.
- Reflexivity: for any x, y, if $x \sim y$, then $f(x) \equiv f(y)$, so $f(y) \equiv f(x)$ and $y \sim x$.
- Transitivity: if $x \sim y \sim z$, then $f(x) \equiv f(y) \equiv f(z)$, so $f(x) \equiv f(z)$ and so $x \sim z$.

The equivalence classes are the pullbacks of the equivalence classes on T.