Note 23

3 Vector Potentials

DEF $\mathbb{A}$ is called a **vector potential** of $\mathbb{F}$ if : $\mathbb{F} = \text{curl}(\mathbb{A})$.

Remark) By a direct computation, we get $\text{div}(\text{curl}(\mathbb{A})) = 0$. So, a necessary condition for a vector field $\mathbb{F}$ to have a vector potential is $\text{div}(\mathbb{F}) = 0$.

Ex (Computing flux using circulation) Let $\mathbb{F}$ have a vector potential

$$\mathbb{A} = \langle z-y,\ x\cos z,\ e^{xy} + z^2 \rangle.$$

Let S : $x^2+y^2+z^2 = 1$, $z \geq 0$, oriented outward.

(1) Compute $\mathbb{F}$.

(2) Compute $\iint_S \mathbb{F}\cdot d\mathbb{S}$.

Sol)

(1) : $\mathbb{F} = \text{curl}(\mathbb{A}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z-y & x\cos z & e^{xy}+z^2 \end{vmatrix} = \left\langle xe^{xy} - x\sin z,\ 1-ye^{xy},\ \cos z + 1 \right\rangle.$

(2) : ∂S : $x^2+y^2 = 1$, $z=0$, oriented CCW viewed from above. So by Stokes' Thm,

$$\iint_S \mathbb{F}\cdot d\mathbb{S} = \iint_S \text{curl}(\mathbb{A})\cdot d\mathbb{S} = \oint_{\partial S} \mathbb{A}\cdot d\mathbf{r} = \oint_{\partial S} \underbrace{-y\,dx + x\,dy + e^{xy}\,dz}_{z=0 \text{ over } \partial S}$$

$$= \int_0^{2\pi} d\theta = \boxed{2\pi}.$$

∎

Ex (Vector Potential of a constant vector field) Let $\mathbb{C} = \langle a, b, c\rangle$ be a constant vector field.

(1) Show that $\mathbb{A} = \frac{1}{2}\mathbb{C}\times\mathbf{r}$ is a vector potential of $\mathbb{C}$, where $\mathbf{r} = \langle x, y, z\rangle$.

(2) If S is a bounded region in the plane $ax+by+cz = d$, then show that

$$\text{area}(S) = \left| \frac{1}{2\|\mathbb{C}\|} \oint_{\partial S} (\mathbb{C}\times\mathbf{r})\cdot d\mathbf{r} \right|$$

Sol) (1) We have

$$\mathbb{A} = \frac{1}{2}\langle c_2 z - c_3 y,\ c_3 x - c_1 z,\ c_1 y - c_2 x\rangle.$$

Then

$$\operatorname{curl}(\mathbb{A}) = \frac{1}{2}\begin{vmatrix} \mathbb{i} & \mathbb{j} & \mathbb{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ bz - cy & cx - az & ay - bx \end{vmatrix}$$

$$= \frac{1}{2}\langle a-(-a),\ b-(-b),\ c-(-c)\rangle = \mathbb{C}.$$

(2) The unit normal $\mathbb{n}$ of S must be $\mathbb{n} = \pm\frac{\mathbb{C}}{\|\mathbb{C}\|}$. So

$$\frac{1}{2}\oint_{\partial S}(\mathbb{C}\times\mathbb{r})\cdot d\mathbb{r} \underset{\text{(Stokes)}}{=} \iint_S (\mathbb{C}\cdot\mathbb{n})\,dS = \iint_S (\pm\|\mathbb{C}\|)\,dS = \pm\|\mathbb{C}\|\,\operatorname{area}(S).$$

Taking absolute value proves the formula. $\square$

Ex (Surface-Independence)

Let $\begin{cases} \mathbb{F} = \langle xy\sin(yz),\ xe^{z}-1,\ \cos(yz)\rangle. \\ S:\ \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + z^2 = 1, \quad z \geq 0, \text{ outward-pointing normal.} \end{cases}$

Suppose we know that $\mathbb{F}$ has a vector potential. Compute $\iint_S \mathbb{F}\cdot d\mathbb{S}$.

Sol) A direct computation seems too complicated, but we may take advantage of the fact that $\mathbb{F}$ has a vector potential:

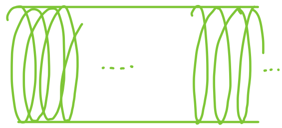
Let T: $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 \leq 1$, $z = 0$, upward-pointing normal. Then $\partial S = \partial T$, and so, by the Stokes' Theorem (or "surface-independence"),

$$\iint_S \mathbb{F}\cdot d\mathbb{S} = \iint_T \mathbb{F}\cdot d\mathbb{S} = \iint_T (\mathbb{F}\cdot\mathbb{k})\,dS = \iint_T 1\,dS = 6\pi.$$

$\square$

<u>Ex</u> (Example 4 in Textbook) In electromagnetics, a megnetic field $\mathbb{B}$ is a primary example of a vector field having a vector potential $\mathbb{A}$.

If an electric current flows through a solenoid,



solenoid : tightly wound spiral of wire

It creates a magnetic field $\mathbb{B}$. If the solenoid is infinitely long cylinder $x^2+y^2=R^2$, then $\mathbb{B}$ takes the form

$$\mathbb{B} = \begin{cases} 0, & r>R \\ B\mathbb{k}, & r<R, \end{cases} \qquad r=\sqrt{x^2+y^2}.$$

One can compute a vector potential $\mathbb{A}$ of $\mathbb{B}$ as

$$\mathbb{A} = \begin{cases} \frac{1}{2}\left(\frac{R}{r}\right)^2 B\langle -y, x, 0\rangle, & \text{if } r>R \\ \frac{1}{2} B\langle -y, x, 0\rangle, & \text{if } r<R \end{cases}$$

(1) If S is any closed surface, then the magnetic flux through S is

$$\iint_S \mathbb{B}\cdot d\mathbb{S} \overset{\text{(Stokes)}}{=} \oint_{\partial S} \mathbb{A}\cdot d\mathbb{r} = 0$$

Because $\partial S = \phi$.

(2) If S is any surface s.t. ∂S : circle of radius $r>R$ in the xy-plane at $\mathbb{0}$, oriented CCW when viewed from above,

$$\iint_S \mathbb{B}\cdot d\mathbb{S} = \oint_{\partial S} \mathbb{A}\cdot d\mathbb{r} = \int_0^{2\pi} \frac{1}{2}\left(\frac{R}{r}\right)^2 B \cdot r^2 dt = \pi R^2 B.$$

$\partial S : \mathbb{r}(t) = \langle r\cos t, r\sin t, 0\rangle, \quad 0 \le t \le 2\pi$