Note 21

1

Section 18.1 Green's Theorem

<u>Recall)</u>
- Boundary orientation
- $\text{curl}_z(\mathbf{F}) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$
- Green's Theorem : If $\mathbf{F}$ is continuously differentiable on D,

$$\oint_{\partial D} F_1\,dx + F_2\,dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA,$$

or in the "circulation form" :

$$\oint_{\partial D} \mathbf{F}\cdot d\mathbf{r} = \iint_D \text{curl}_z(\mathbf{F})\, dA.$$

$\boxed{3}$ Area

- By the Green's Theorem,

$$\text{area}(D) = \iint_D 1\,dxdy = \oint_{\partial D} x\,dy = -\oint_{\partial D} y\,dx = \frac{1}{2}\oint_{\partial D} x\,dy - y\,dx.$$

<u>Ex</u> Compute the area of D enclosed by $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ using Green's Thm.

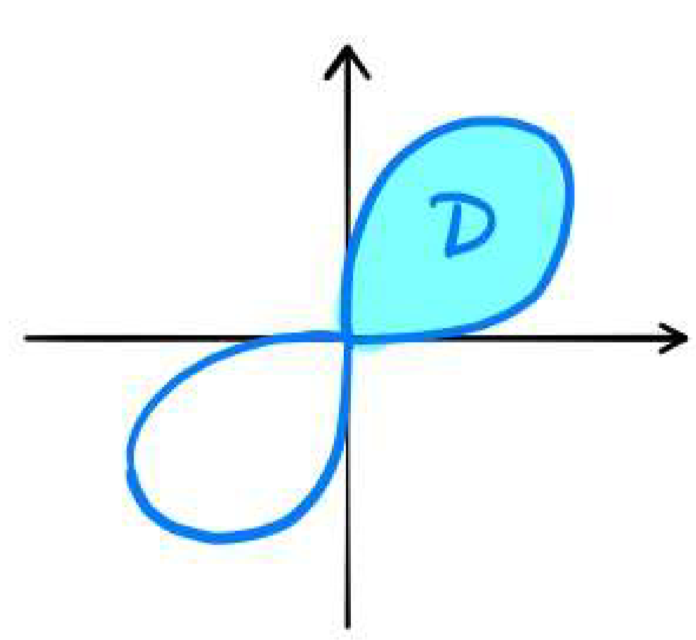
<u>Sol)</u> Parametrize ∂D by $(x, y) = (a\cos\theta,\ b\sin\theta)$, $0 \le \theta \le 2\pi$. Then

$$\text{area}(D) = \frac{1}{2}\oint_{\partial D} x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} \left((a\cos\theta)(b\cos\theta) - (b\sin\theta)(-a\sin\theta)\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab\,d\theta = ab\pi.$$

$\square$

<u>Ex</u> (Lemniscate) Compute the area enclosed by $(x^2+y^2)^2 = xy$, $x, y \ge 0$.



<u>Sol)</u> In polar coordinates, the equation becomes $r = f(\theta) = \sqrt{\cos\theta \sin\theta}$. Then we may parametrize the curve by

$$(x, y) = (f(\theta)\cos\theta,\ f(\theta)\sin\theta), \qquad 0 \le \theta \le \frac{\pi}{2}.$$

which gives

$$\begin{aligned} x\,dy - y\,dx &= f(\theta)\cos\theta\,\big(f'(\theta)\sin\theta + f(\theta)\cos\theta\big)\,d\theta \\ &\quad - f(\theta)\sin\theta\,\big(f'(\theta)\cos\theta - f(\theta)\sin\theta\big)\,d\theta \\ &= f(\theta)^2\,d\theta. \end{aligned}$$

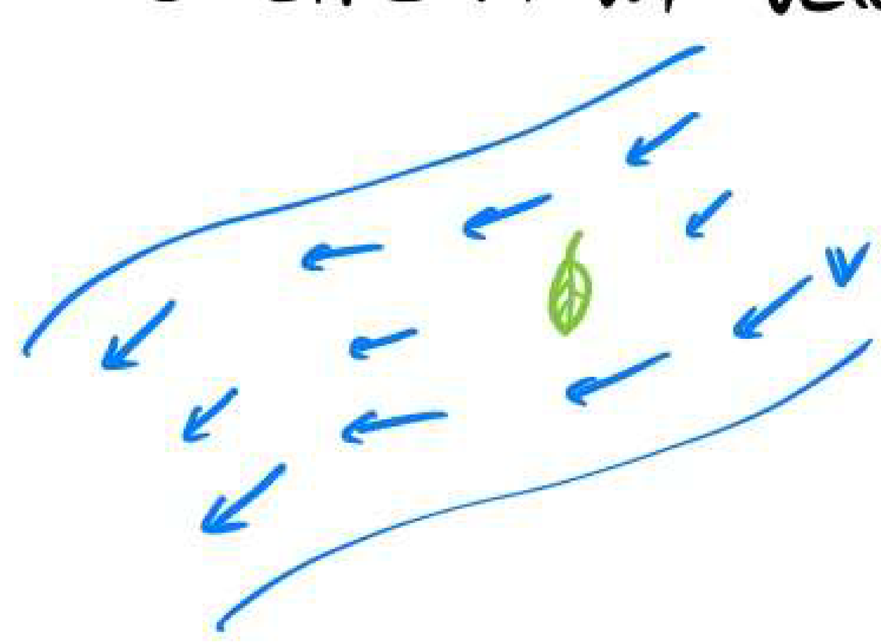
So we have

$$\begin{aligned} \text{area}(D) &= \frac{1}{2}\oint_{\partial D} x\,dy - y\,dx \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2}\cos\theta\sin\theta\,d\theta = \left[\frac{1}{4}\sin^2\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{4}. \end{aligned}$$

∎

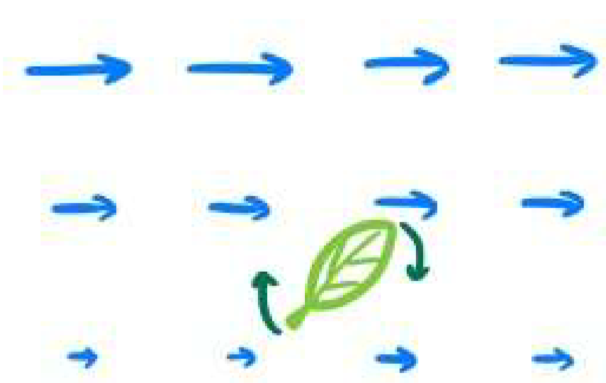
4 Physical Meaning of Curl

- Imagine a small leaf floating on a stream w/ velocity vector field $\mathbb{V}$.

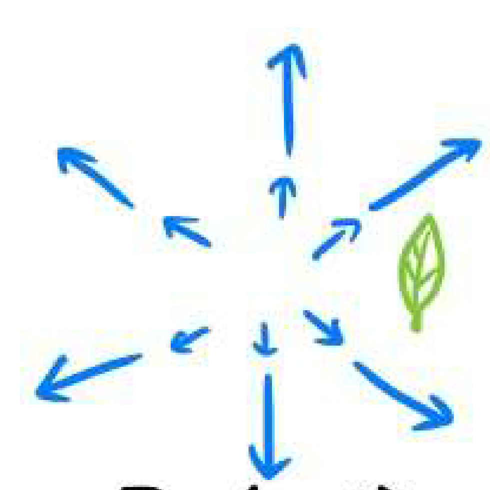


<u>Ex</u>

$\mathbb{F} = \langle 0, x \rangle$
$\text{curl}_z(\mathbb{F}) = 1$



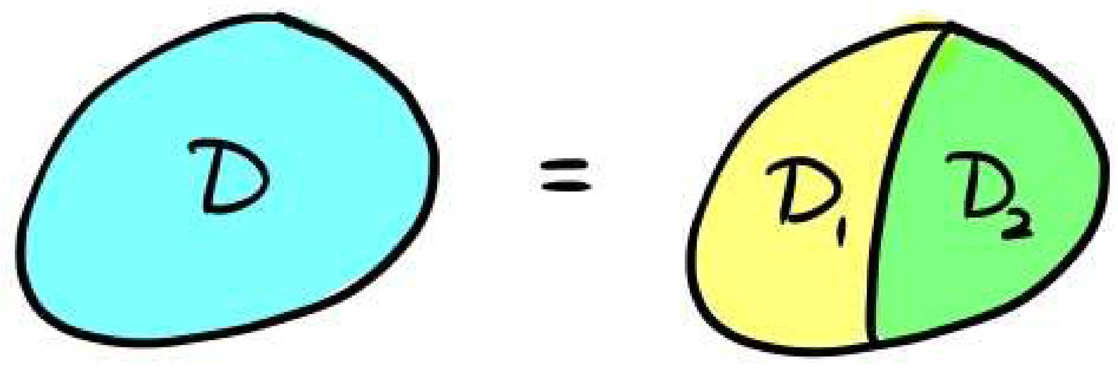
$\mathbb{F} = \langle y, 0 \rangle$,
$\text{curl}_z(\mathbb{F}) = -1$



$\mathbb{F} = \langle x, y \rangle$
$\text{curl}_z(\mathbb{F}) = 0$

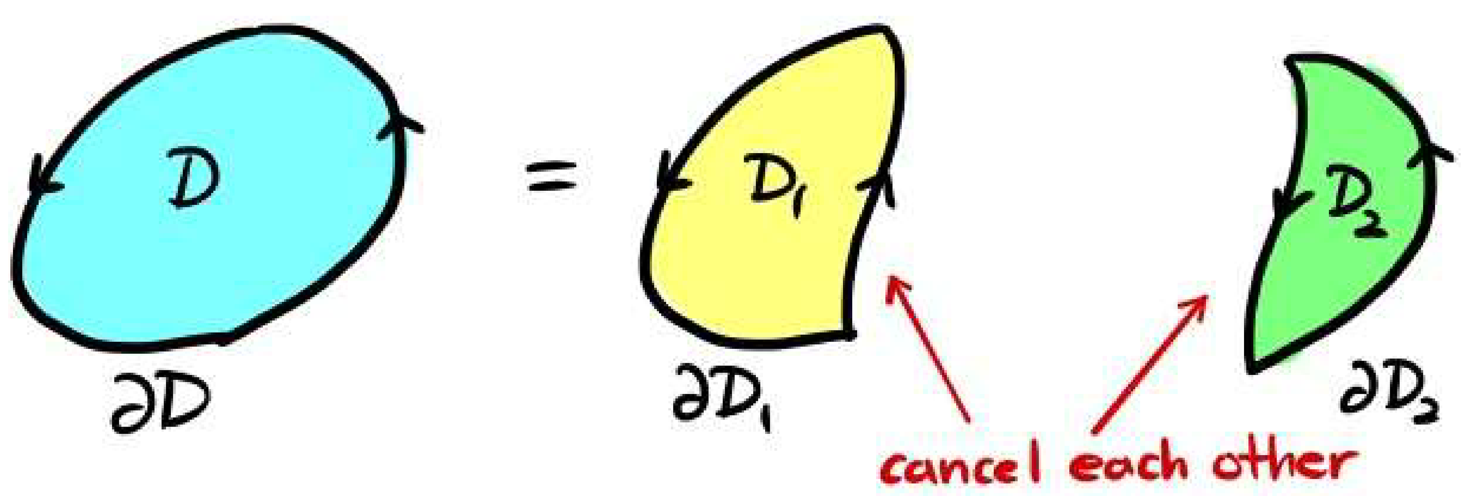
5 Additivity of Circulation

- If $D = D_1 \cup D_2$, where D_1 & D_2 : non-overlapping,



then we have :

$$\oint_{\partial D} \mathbb{F} \cdot d\mathbf{r} = \oint_{\partial D_1} \mathbb{F} \cdot d\mathbf{r} + \oint_{\partial D_2} \mathbb{F} \cdot d\mathbf{r}$$



This is because $\partial D = \partial D_1 + \partial D_2$.

Ex If $\begin{cases} \mathbb{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle : \text{vortex field}, \\ D : \text{region s.t. } (0,0) \notin \partial D. \end{cases}$

Then

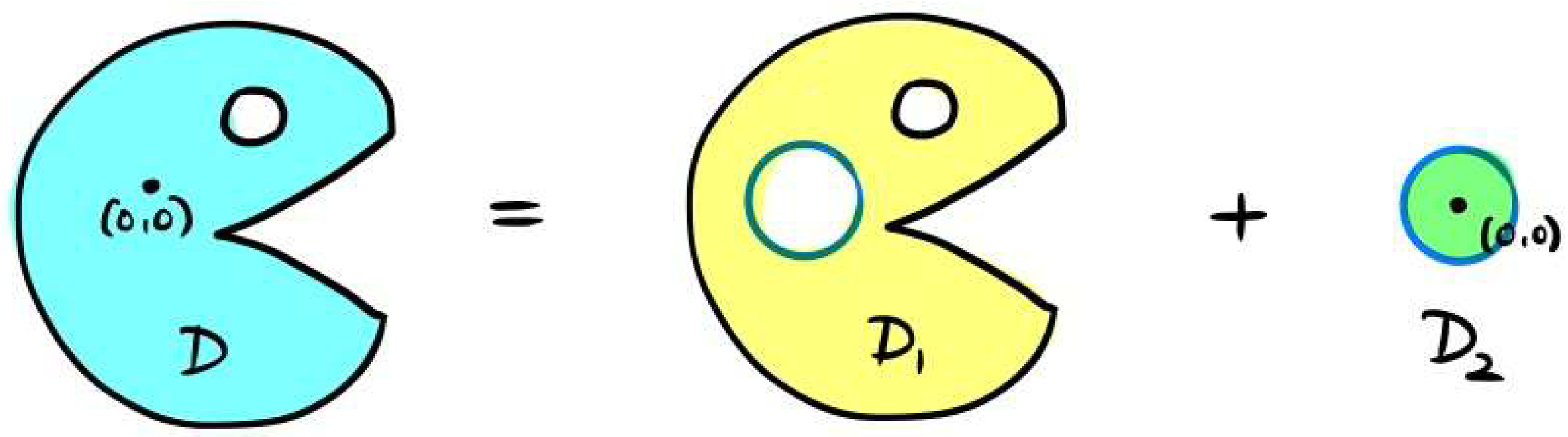
$$\oint_{\partial D} \mathbb{F} \cdot d\mathbf{r} = \begin{cases} 0, & \text{if } (0,0) \notin D \\ 2\pi, & \text{if } (0,0) \in D. \end{cases}$$

Pf) (1) If $(0,0) \notin D$, then Green's Thm is applicable :

$$\oint_{\partial D} \mathbb{F} \cdot d\mathbf{r} = \iint_D \underbrace{\text{curl}_z(\mathbb{F})}_{=0} \, dA = 0.$$

(2) If $(0,0) \in D$, then $\mathbb{F}$ is undefined at $(0,0)$, so Green's Thm is not directly applicable.

But, writing



So that ∂D_2 : circle of radius R at 0, then

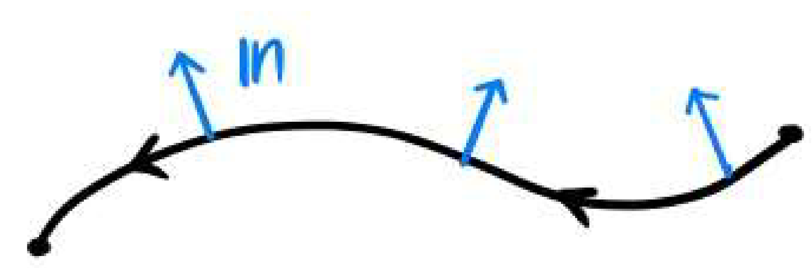
- ▷ $\oint_D \mathbb{F} \cdot d\mathbf{r} = \oint_{\partial D_1} \mathbb{F} \cdot d\mathbf{r} + \oint_{\partial D_2} \mathbb{F} \cdot d\mathbf{r}$
- ▷ $\oint_{\partial D_1} \mathbb{F} \cdot d\mathbf{r} = 0$ as in (1) $\Rightarrow$ $\oint_{\partial D} \mathbb{F} \cdot d\mathbf{r} = 2\pi$.
- ▷ $\oint_{\partial D_2} \mathbb{F} \cdot d\mathbf{r} = 2\pi$ by direct computation.

$\square$

$\boxed{6}$ Flux Form of Green's Theorem

Recall) • If C : oriented curve, then

$$\mathbf{n} = \left[\begin{array}{c} \text{unit normal vector obtained by} \\ \text{rotating } \mathbb{T} \text{ by } -90^\circ. \end{array} \right]$$



• Flux of $\mathbb{F}$:

$$\int_C (\mathbb{F} \cdot \mathbf{n})\, ds = \int_C F_1\, dy - F_2\, dx.$$

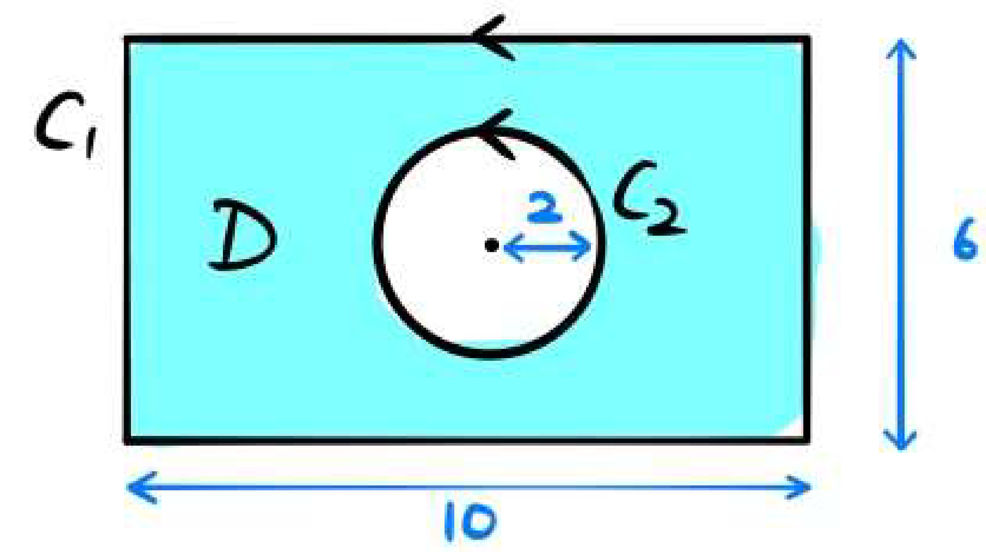
• $\operatorname{div}(\mathbb{F}) = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y}$.

THM If $\mathbb{F}$ is conti. diff-ble on D, then

$$\oint_{\partial D} (\mathbb{F} \cdot \mathbf{n})\, ds = \iint_D \operatorname{div}(\mathbb{F})\, dA.$$

⊡ More Examples

Ex Let C_1, C_2, D be as follows:



Suppose: $\oint_{C_2} \mathbb{F} \cdot d\mathbb{r} = 12$, $\quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -3$ in D.

Compute $\oint_{C_1} \mathbb{F} \cdot d\mathbb{r}$.

Sol) $\partial D = C_1 - C_2$, or equivalently, $C_1 = \partial D + C_2$. So

$$\oint_{C_1} \mathbb{F} \cdot d\mathbb{r} = \oint_{\partial D} \mathbb{F} \cdot d\mathbb{r} + \underbrace{\oint_{C_2} \mathbb{F} \cdot d\mathbb{r}}_{=12}$$

$$\overset{(\text{Green})}{=} \iint_D \underbrace{\mathrm{curl}_z(\mathbb{F})}_{=-3} \, dA + 12$$

$$= -3 \, \mathrm{area}(D) + 12$$

$$= -3(60 - 4\pi) + 12$$

$$= 12\pi - 168.$$

$\square$