Note 20

Chapter 18. Fundamental Theorems of Vector Analysis

BIG PICTURE

1D

$$\begin{tikzcd}
F \arrow[r, "\frac{d}{dx}"] \arrow[d, dash] & f \arrow[d, dash] \\
\text{“0-dim integral”} \; F(b)-F(a) \arrow[r, equal, "\text{FToC}"] \arrow[d, dash] & \text{integral} \; \int_a^b f(x)\,dx \arrow[d, dash] \\
\text{“oriented points”} \; \underset{a}{\ominus} \; \underset{b}{\oplus} & \text{interval } [a,b] \arrow[l, "\text{“boundary } \partial\text{”}"]
\end{tikzcd}$$

2D

$$\begin{tikzcd}
f \arrow[r, "\text{grad}"] \arrow[rr, bend left, "\text{curl}_z(\text{grad}(f)) = 0"] \arrow[d, dash] & \mathbb{F} \arrow[r, "\text{curl}_z\ (?)"] \arrow[d, dash] & g \arrow[d, dash] \\
\text{0-dim integral} \arrow[r, equal, "\text{FT for conservative VF}"] \arrow[d, dash] & \text{vector line integral} \arrow[r, equal, "?"] \arrow[d, dash] & \text{double integral} \arrow[d, dash] \\
\text{oriented points} & \text{oriented curve} \arrow[l, "\partial"] & \text{planar region} \arrow[l, "\partial"]
\end{tikzcd}$$

3D

$$\begin{tikzcd}
f \arrow[r, "\text{grad}"] \arrow[rr, bend left, "\text{curl}(\text{grad}(f)) = 0"] \arrow[d, dash] & \mathbb{F} \arrow[r, "\text{curl}"] \arrow[rr, bend left, "\text{div}(\text{curl}(\mathbb{F})) = 0"] \arrow[d, dash] & \mathbb{G} \arrow[r, "\text{div}"] \arrow[d, dash] & g \arrow[d, dash] \\
\text{0-dim integral} \arrow[r, equal, "\text{FT for conservative VF}"] \arrow[d, dash] & \text{vector line integral} \arrow[r, equal, "?"] \arrow[d, dash] & \text{vector surface integral} \arrow[r, equal, "?"] \arrow[d, dash] & \text{triple integral} \arrow[d, dash] \\
\text{oriented points} & \text{oriented curve} \arrow[l, "\partial"] & \text{oriented surface} \arrow[l, "\partial"] & \text{solid region} \arrow[l, "\partial"]
\end{tikzcd}$$

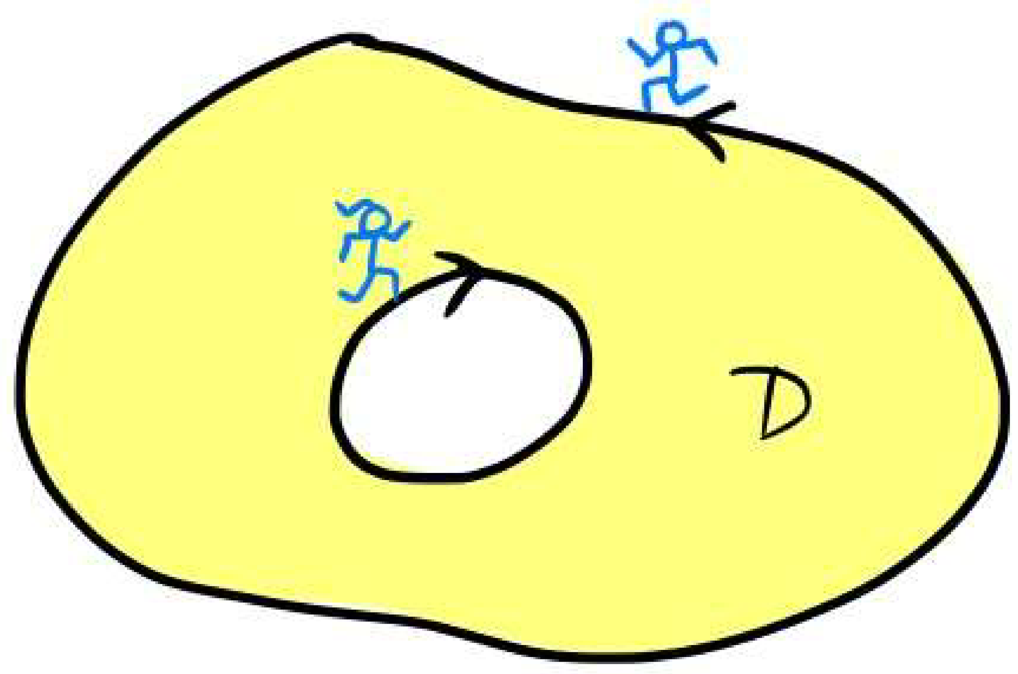
Section 18.1 Green's Theorem

1 Setting & Statement

DEF If D : domain in $\mathbb{R}^2$, then

(1) ∂D : boundary of D.

(2) Unless stated otherwise, ∂D assumes boundary orientation : the traversing direction s.t. D lies always left :



DEF $[\text{curl}_z \text{ of } \langle F_1, F_2 \rangle] = \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}$.

THM (Green's Theorem) Let D : domain in $\mathbb{R}^2$ with "piecewise smooth" boundary with boundary orientation. Then

$$\oint_{\partial D} \mathbb{F} \cdot d\mathbf{r} = \iint_D \text{curl}_z(\mathbb{F})\, dx\,dy.$$

I.e.,

$$\oint_{\partial D} F_1\, dx + F_2\, dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy.$$

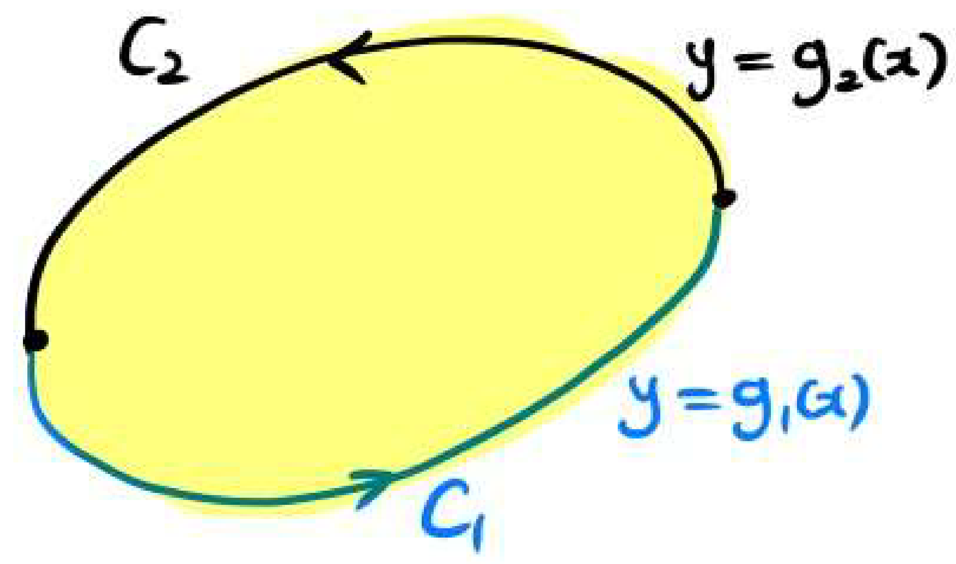
- The proof for the general case is quite involved, so we only prove for a special case and explain how this generalizes.

Pf, Special Case) Suppose D : both vertically / horizontally simple.

Then

① As a vertically simple region, may write

$$D: \quad a \le x \le b, \quad g_1(x) \le y \le g_2(x)$$



Then we may parametrize:

$$C_1 : \mathbf{r}(t) = \langle t, g_1(t) \rangle, \quad a \le t \le b$$
$$-C_2 : \mathbf{r}(t) = \langle t, g_2(t) \rangle, \quad a \le t \le b.$$

Then

$$\oint_{\partial D} F_1\, dx = \int_{C_1} F_1\, dx - \int_{-C_2} F_1\, dx$$
$$= \int_a^b F_1(t, g_1(t))\, dt - \int_a^b F_1(t, g_2(t))\, dt$$
$$= -\int_a^b \left(F_1(t, g_2(t)) - F_1(t, g_1(t)) \right) dt$$
$$= -\int_a^b \int_{g_1(t)}^{g_2(t)} \frac{\partial F_1}{\partial y}(t, y)\, dy\, dt$$
$$= -\iint_D \frac{\partial F_1}{\partial y}\, dx\, dy.$$

A similar argument shows

$$\oint_{\partial D} F_2\, dy = \iint_D \frac{\partial F_2}{\partial x}\, dx\, dy.$$

Adding two results proves the theorem. $\square$

$\boxed{2}$ Examples

Ex $\oint_{\partial D} y^2 dx + x^2 dy$, $\quad D = [0,1] \times [0,1]$.

$$= \iint_D (2x - 2y)\, dx dy = 0 \quad \text{by symmetry.}$$

$\square$

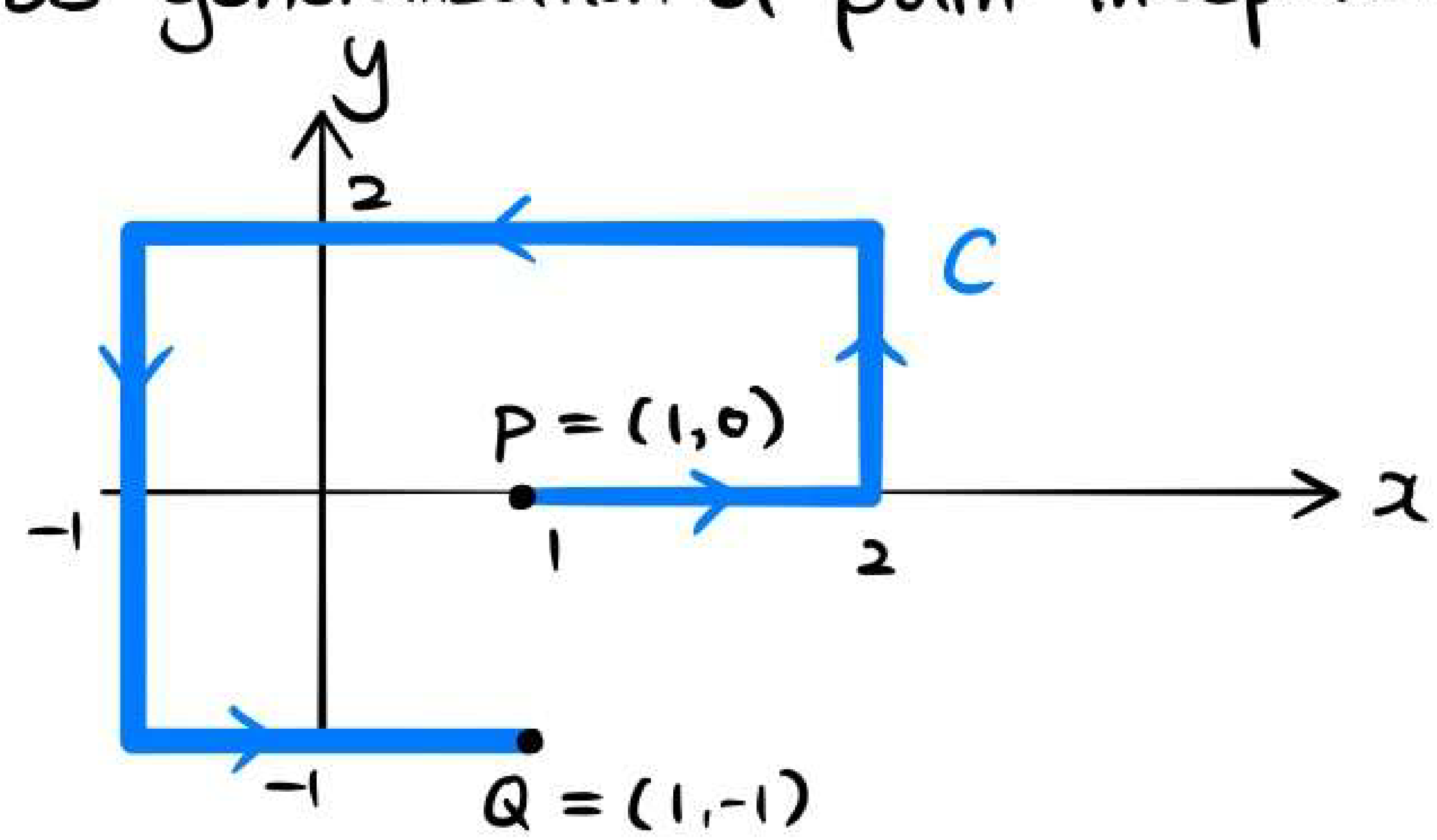
Ex $\oint_C x^2 y\, dx$, $\quad C$: unit circle at the origin, CCW oriented.

Sol) Realize C as ∂D for D : unit disk at 0. Then

$$\oint_C x^2 y\, dx = \oint_{\partial D} x^2 y\, dx = \iint_D -x^2\, dx dy = -\int_0^{2\pi}\int_0^1 r^3 \cos^2\theta\, dr d\theta = -\frac{\pi}{4}.$$

$\square$

Ex (Green's Thm as generalization of path-independence) Let C be as follows:



Compute $\oint_C \mathbb{F} \cdot d\mathbb{r}$ for $\mathbb{F}(x,y) = \langle x^3, 4x \rangle$.

Sol) Let C_2 = line seg. from P to Q. Then

$$\int_C \mathbb{F} \cdot d\mathbb{r} = \int_{C_2} \mathbb{F} \cdot d\mathbb{r} + \int_{C - C_2} \mathbb{F} \cdot d\mathbb{r}$$

Realize $C - C_2$ as the ∂ of some domain D (enclosed by $C - C_2$). Then

$$= \int_{C_2} \mathbb{F} \cdot d\mathbb{r} + \iint_D \underbrace{\text{curl}_z(\mathbb{F})}_{=4}\, dA.$$

Compute this!

$\square$

3 Area

- By the Green's Theorem,

$$\text{area}(D) = \iint_D 1\, dx\,dy = \oint_{\partial D} x\, dy = -\oint_{\partial D} y\, dx = \frac{1}{2}\oint_{\partial D} x\,dy - y\,dx.$$

Ex Compute the area of D enclosed by $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ using Green's Thm.

Sol) Parametrize ∂D by $(x,y) = (a\cos\theta,\ b\sin\theta)$, $0 \le \theta \le 2\pi$. Then

$$\text{area}(D) = \frac{1}{2}\oint_{\partial D} x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} \big((a\cos\theta)(b\cos\theta) - (b\sin\theta)(-a\sin\theta)\big)\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab\, d\theta = ab\pi.$$

$\square$