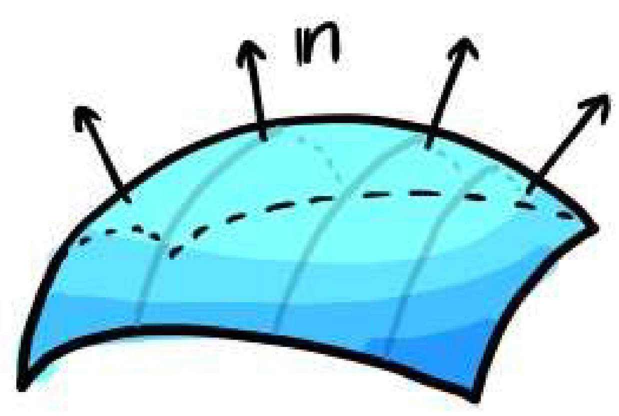
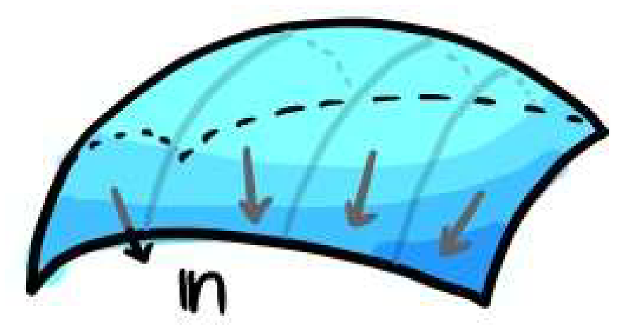
Section 17.5 Surface Integrals of Vector Fields

[1] Oriented Surfaces and Vector Surface Integrals

DEF • An orientation of a surface S is a choice of unit normal vectors $\mathbb{n}(P)$ at each $P \in S$ in a continuous way.





2 possible orientations on S

• Oriented surface = [surface] + [choice of orientation].

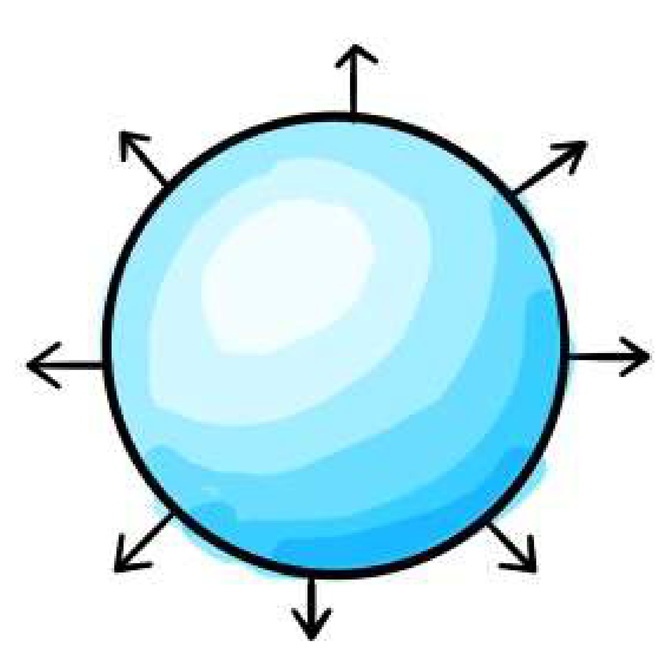
DEF Let S : surface w/ orientation $\mathbb{n}$,
$\mathbb{F}$: vector field,
then

• [normal component at P] $:= \mathbb{F}(P) \cdot \mathbb{n}(P)$

• [vector surface integral] $:= \iint_S (\mathbb{F} \cdot \mathbb{n})\, dS$, a.k.a. the flux of $\mathbb{F}$ across S.

Remark A vector surface integral depends on the orientation!

Ex Let S: sphere $x^2+y^2+z^2=R^2$, oriented outward. Then the flux of $\mathbb{F}$, given by:



$$\mathbb{F} = \nabla \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$= \left\langle -\frac{x}{(x^2+y^2+z^2)^{3/2}},\ -\frac{y}{(x^2+y^2+z^2)^{3/2}},\ -\frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

across S is

$$\iint_S (\mathbb{F}\cdot\mathbb{n})\,dS = \iint_S \left(\left(-\frac{\mathbb{n}}{R^2}\right)\cdot\mathbb{n}\right) dS = \iint_S \left(-\frac{1}{R^2}\right) dS = -4\pi.$$

$\mathbb{F} = -\frac{\mathbb{n}}{R^2}$ on S.

$\square$

$\boxed{2}$ Parametric Form

DEF Let S : surface w/ orientation $\mathbb{n}$

G : parametrization of S.

Then G has $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$ orientation if $\dfrac{\mathbb{N}}{\|\mathbb{N}\|} = \left\{\begin{matrix}\mathbb{n},\\ -\mathbb{n}.\end{matrix}\right.$

In other words, S and G have the $\left\{\begin{matrix}\text{same}\\ \text{opposite}\end{matrix}\right\}$ orientations.

- If G has $+$ orientation, then

$$(\mathbb{F}\cdot\mathbb{n})\,dS = \left(\mathbb{F}\cdot\frac{\mathbb{N}}{\cancel{\|\mathbb{N}\|}}\right)\cancel{\|\mathbb{N}\|}\,du\,dv = \mathbb{F}\cdot\mathbb{N}\,du\,dv$$

- In this regard, we define

$$[\text{vector surface differential}] = d\mathbf{S} = \mathbb{n}\,dS.$$

THM Let S : oriented surface,

G : parametrization from D to S, positively oriented,

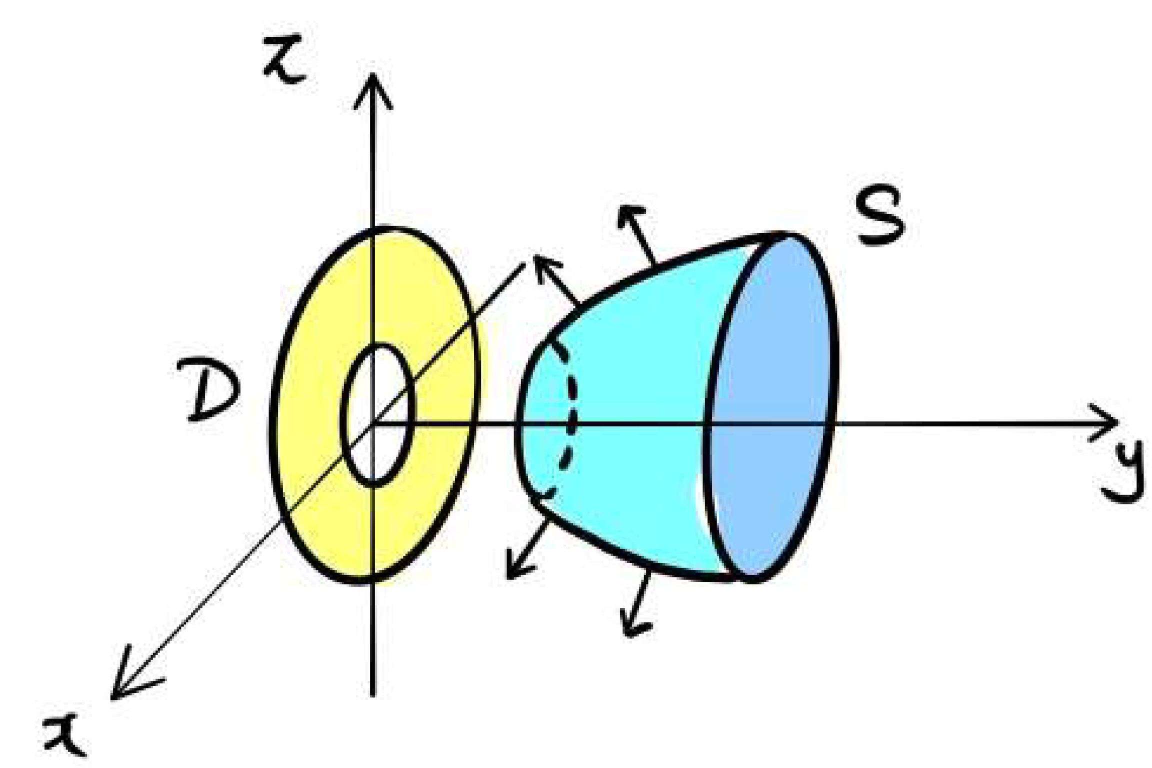
Assume G is "smooth and nonoverlapping". Then

$$\iint_S \mathbb{F}\cdot d\mathbf{S} = \iint_S (\mathbb{F}\cdot\mathbb{n})\,dS = \iint_D \mathbb{F}(G(u,v))\cdot\mathbb{N}(u,v)\,du\,dv.$$

Ex Compute the integral of $\mathbb{F} = x^2 \mathbf{j}$ over the surface

$$S : \quad y = x^2 + z^2 \quad \text{for } 1 \le y \le 4,$$

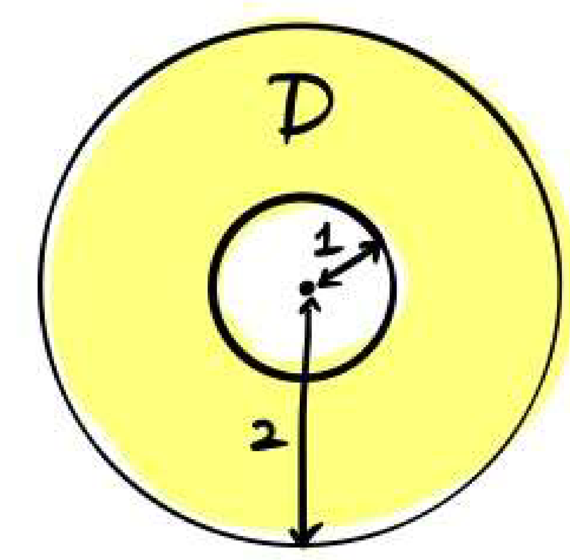
oriented in the negative y-direction.



Sol) • Step 1. Find the parametrization:

We may consider S as the graph of the function x^2+z^2 over

$$D = \{(x,z) : \ 1 \le x^2+y^2 \le 4\}$$



Then we may set

$$G(x,z) = (x, \ x^2+y^2, \ z).$$

(At this point, it is not clear if G is positively oriented.)

• Step 2. Compute $\mathbb{N}$:

$$\mathbb{T}_x = \langle 1, \ 2x, \ 0 \rangle$$

$$\mathbb{T}_z = \langle 0, \ 2z, \ 1 \rangle$$

$$\mathbb{N} = \langle 2x, -1, 2z \rangle$$

G is indeed pos. oriented!

• Step 3. Evaluate the integral:

$$\iint_S \mathbb{F} \cdot d\mathbb{S} = \iint_D \langle 0, x^2, 0 \rangle \cdot \langle 2x, -1, 2z \rangle \, dx\,dz$$

$$= -\iint_D x^2 \, dx\,dz.$$

We may then invoke polar coordinates to write:

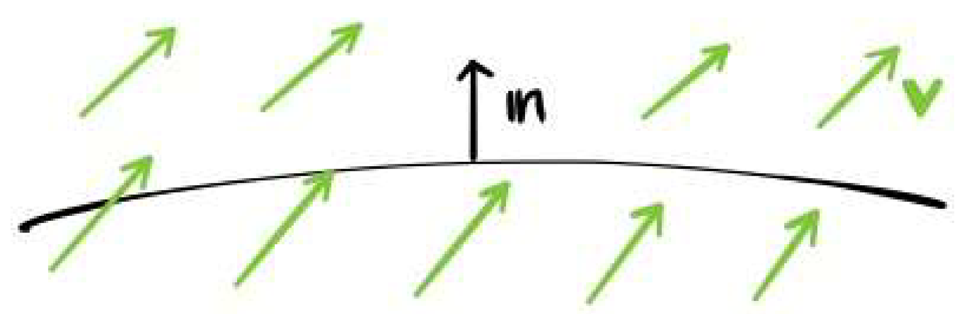
$$= -\int_0^{2\pi}\int_1^2 (r\cos\theta)^2 \cdot r\,dr\,d\theta$$

$$= -\pi \cdot \frac{2^4 - 1^4}{4} = -\frac{15}{4}\pi.$$

$\square$

3 Fluid Flux

- Suppose $\begin{cases} \mathbb{V} : \text{velocity vector field of fluid} \\ S : \text{surface} \end{cases}$



- Assume : a fluid particle of dimensions $\Delta S \times \ell$ has crossed the surface within Δt time interval :

: this fluid particle has travelled $(\mathbb{V}\Delta t)$ distance.

Then we must have $\ell = |(\mathbb{V}\Delta t)\cdot \mathbb{n}| = |\mathbb{V}\cdot\mathbb{n}|\,\Delta t$. Moreover, the sign of $\mathbb{V}\cdot\mathbb{n}$ determines whether the particle moves along or against the direction of $\mathbb{n}$:

$\operatorname{sign}(\mathbb{V}\cdot\mathbb{n}) = +1$ $\qquad\qquad$ $\operatorname{sign}(\mathbb{V}\cdot\mathbb{n}) = -1$

- Altogether, the net amount of flow that passes through the oriented surface S within Δt time interval is approximately

$$\sum (\mathbb{V}\cdot\mathbb{n})\,\Delta t\,\Delta S \approx \left(\iint_S \mathbb{V}\cdot d\mathbb{S}\right)\Delta t.$$

$\therefore$ [Flow rate across the S] $= \iint_S \mathbb{V}\cdot d\mathbb{S}$.

<u>Ex</u> If $\mathbb{V} : \langle x, y, e^z \rangle$: velocity vector field,

S : cylinder $x^2+y^2=4$, $1 \le z \le 3$, oriented outward,

then the flow rate across S is

$$\iint_S \mathbb{V}\cdot d\mathbb{S} = \iint_S (\mathbb{V}\cdot\mathbb{n})\, dS$$

$\mathbb{n} = \frac{1}{2}\langle x, y, 0\rangle$, either by computation or by geometric insight

$$= \iint_S \mathbb{V}\cdot \tfrac{1}{2}\langle x, y, 0\rangle\, dS$$

$$= \iint_S \tfrac{1}{2}(x^2+y^2)\, dS$$

$x^2+y^2 = 4$ on S

$$= \iint_S 2\, dS = 2\cdot \underbrace{8\pi}_{\text{area of } S} = 16\pi.$$

□

⊡ Extra

- Time permitting, do the following problems:

<u>Ex</u> (Beware of the orientation!) Compute $\iint_S \mathbb{F}\cdot d\mathbb{S}$ for

$$\begin{cases} \mathbb{F} = \langle xy, y, 0\rangle, \\ S : \text{cone } z^2 = x^2+y^2,\quad x^2+y^2 \le 4,\quad z \ge 0, \text{ downward-pointing normal.} \end{cases}$$

<u>Ex</u> (Piecewise-smooth surface?) Compute $\iint_S \mathbb{F}\cdot d\mathbb{S}$ for

$$\begin{cases} \mathbb{F} = \langle 0, 0, e^{y+z}\rangle, \\ S : \text{boundary of the unit cube } [0,1]^3, \text{ outward-pointing normal.} \end{cases}$$