Note 18

Recall

- $\mathbb{T}_u = \frac{\partial G}{\partial u}$, $\quad \mathbb{T}_v = \frac{\partial G}{\partial v}$: tangent vectors
- $\mathbb{N} = \mathbb{T}_u \times \mathbb{T}_v$: normal vector
- Surface integral of $f(x,y,z)$ over S,
$$\iint_S f(x,y,z)\, dS,$$
is defined by the same token as the double integral. In particular,
$$\text{area}(S) = \iint_S 1\, dS.$$
- If $G : D \to S$ is "smooth and non-overlapping", then
$$\iint_S f(x,y,z)\, dS = \iint_D f(G(u,v))\, \|\mathbb{N}(u,v)\|\, du dv.$$

Ex 1 Let $G(x,y) = (x, y, xy)$ over $D = \{(x,y) : x^2+y^2 \leq 1,\ x \geq 0,\ y \geq 0\}$. Then

$$\mathbb{T}_x = \frac{\partial G}{\partial x} = (1, 0, y),$$
$$\mathbb{T}_y = \frac{\partial G}{\partial y} = (0, 1, x),$$
$$\mathbb{N}(x,y) = \mathbb{T}_x \times \mathbb{T}_y = (-y, -x, 1).$$

▷ So, if $S = G(D)$, then

$$\text{area}(S) = \iint_S 1\, dS = \iint_D \|\mathbb{N}(x,y)\|\, dxdy = \iint_D \sqrt{1+x^2+y^2}\, dxdy.$$

This may be computed using polar coordinates:

$$= \int_0^{\frac{\pi}{2}}\int_0^1 \sqrt{1+r^2} \cdot r dr d\theta = \frac{\pi}{2} \cdot \left[\frac{1}{3}(1+r^2)^{3/2} \right]_{r=0}^{r=1} = \frac{\pi}{6}(2\sqrt{2}-1).$$

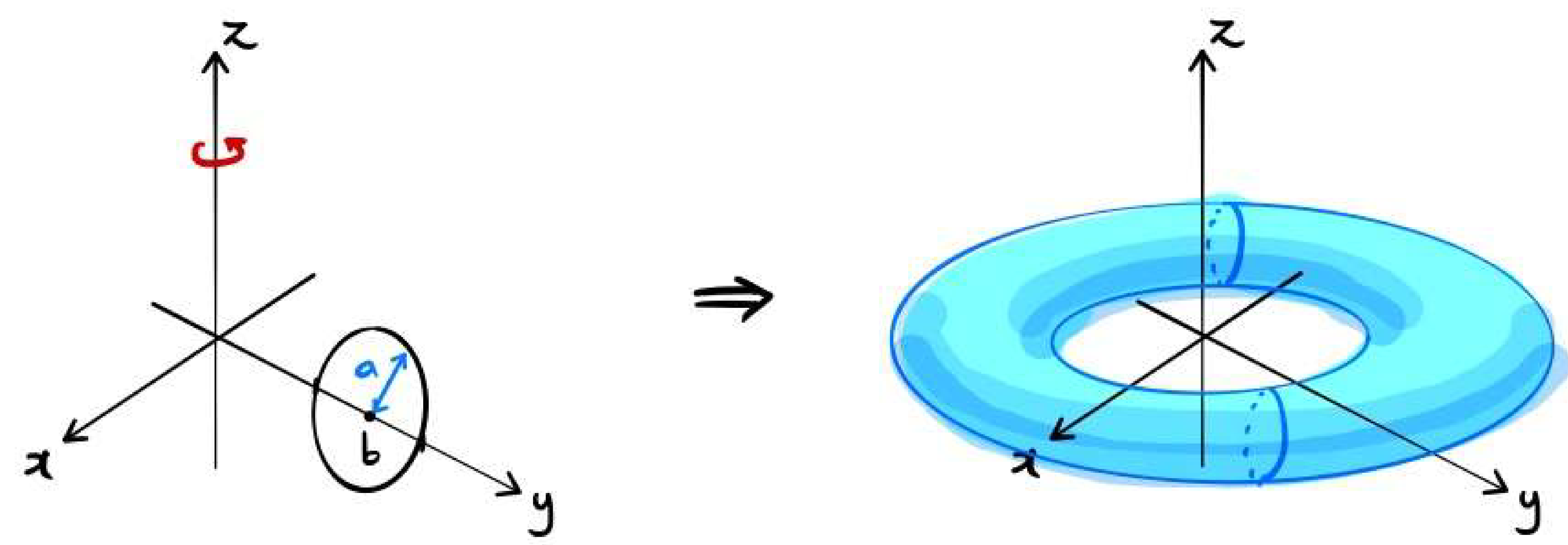
▷ Similarly, the integral of $f(x,y,z) = z$ over S is

$$\iint_S z\, dS = \iint_D xy\sqrt{1+x^2+y^2} = \int_0^{\frac{\pi}{2}}\int_0^1 r^3 \cos\theta \sin\theta \sqrt{1+r^2}\, dr d\theta$$

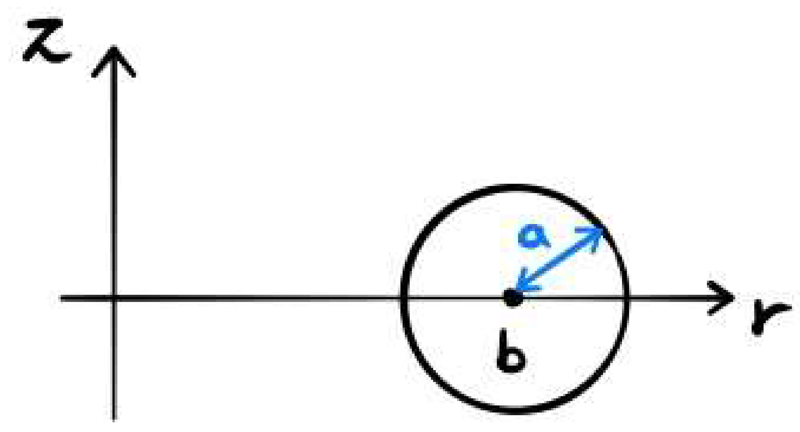
$$= \left[\frac{1}{2}\sin^2\theta \right]_0^{\frac{\pi}{2}} \cdot \left[\frac{(3r^2-2)(r^2+1)^{3/2}}{15} \right]_0^1 = \frac{1+\sqrt{2}}{15}.$$

$\square$

Ex 2 Compute the area of the torus S obtained by rotating the circle in the yz-plane $(y-b)^2+z^2=a^2$ about the z-axis. $(b>a>0)$



Sol) Step 1. Parametrize S : Using the cylindrical coordinates,



$$\begin{cases} r = b + a\cos t \\ z = a\sin t \end{cases} \qquad (0 \le t \le 2\pi)$$

In terms of rectangular coordinates, $(x, y, z) = G(\theta, t)$ is given by

$$\begin{cases} x = r\cos\theta = (b+a\cos t)\cos\theta \\ y = r\sin\theta = (b+a\cos t)\sin\theta \\ z = a\sin t. \end{cases} \qquad 0 \le \theta \le 2\pi, \quad 0 \le t \le 2\pi.$$

Step 2. Compute $\mathbb{T}_\theta$, $\mathbb{T}_t$, $\mathbb{N}(\theta, t)$.

$$\mathbb{T}_\theta = \frac{\partial G}{\partial \theta} = (b+a\cos t)\langle -\sin t, \cos t, 0\rangle.$$

$$\mathbb{T}_t = \frac{\partial G}{\partial t} = a\langle -\sin t\cos\theta, -\sin t\sin\theta, \cos t\rangle.$$

$$\mathbb{N}(\theta, t) = \mathbb{T}_\theta \times \mathbb{T}_t = a(b+a\cos t)\langle \cos t\cos\theta, \cos t\sin\theta, \sin t\rangle.$$

Step 3. Calculate the surface area:

$$\text{area}(S) = \iint_S 1\, dS = \int_0^{2\pi}\int_0^{2\pi} \|\mathbb{N}(\theta, t)\|\, d\theta\, dt$$

$$= \int_0^{2\pi}\int_0^{2\pi} a(a+b\cos t)\, d\theta\, dt = 4\pi^2 ab.$$

□

Ex3 (CoV revisited) Suppose S is a surface lying in the xy-plane and $G : D \to S$ parametrizes S. Then G must take the form

$$G(u,v) = (x(u,v),\ y(u,v),\ 0)$$

Then

$$\mathbb{T}_u = \left\langle \frac{\partial x}{\partial u},\ \frac{\partial y}{\partial u},\ 0 \right\rangle$$
$$\mathbb{T}_v = \left\langle \frac{\partial x}{\partial v},\ \frac{\partial y}{\partial v},\ 0 \right\rangle$$
$$\mathbb{N} = \left\langle 0,\ 0,\ \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right\rangle = \frac{\partial(x,y)}{\partial(u,v)}\,\mathbb{k}.$$

So,

$$\iint_S f\, dS = \iint_D f(G(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv,$$

which is exactly the Change of Variables Formula in 2D! $\square$

Ex 4 (Graph) The surface given by

$$z = g(x,y), \qquad (x,y) \in D$$

can be parametrized as

$$G(x,y) = (x,\ y,\ g(x,y)).$$

Then

$$\mathbb{T}_x = \left\langle 1,\ 0,\ \frac{\partial g}{\partial x} \right\rangle$$
$$\mathbb{T}_y = \left\langle 0,\ 1,\ \frac{\partial g}{\partial y} \right\rangle$$
$$\mathbb{N} = \left\langle -\frac{\partial g}{\partial x},\ -\frac{\partial g}{\partial y},\ 1 \right\rangle$$
$$\Rightarrow \|\mathbb{N}\| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}.$$

For instance, let

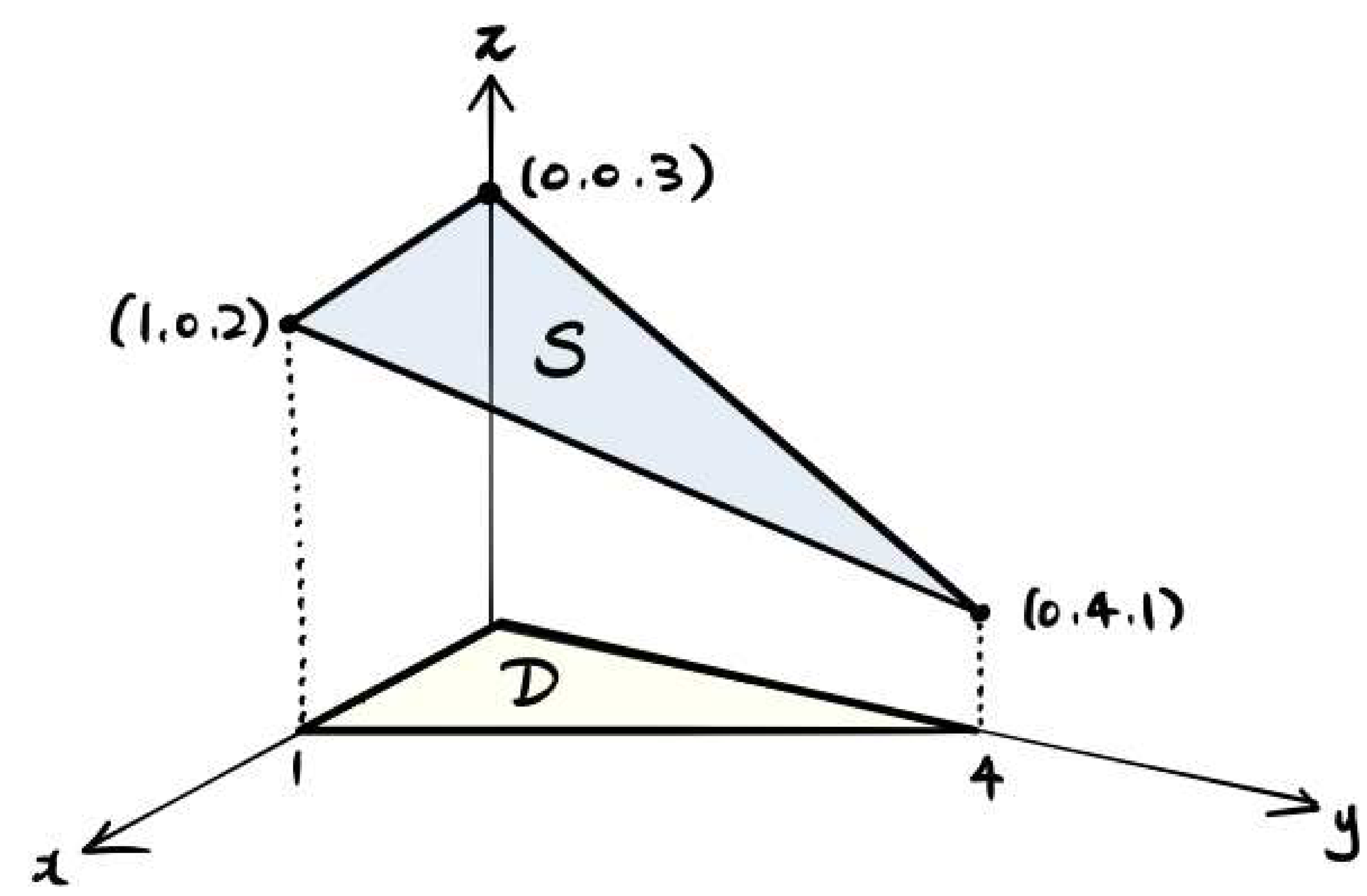
S : triangle joining $(1,0,2)$, $(0,4,1)$, $(0,0,3)$.

This surface is realized as the graph

$$z = g(x,y) = 3 - x - \frac{1}{2}y$$

Over the domain

$$D = \left\{ (x,y) \;:\; \begin{matrix} x \geq 0, \; y \geq 0, \; \text{and} \\ x + \frac{1}{4}y \leq 1 \end{matrix} \right\}$$



Then the integral of $f(x,y,z) = xy + e^z$ over S can be computed by:

$$\iint_S (xy + e^z)\, dS = \iint_D \left(xy + e^{g(x,y)}\right)\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\; dx dy$$

$$= \int_0^4 \int_0^{1-\frac{y}{4}} \left(xy + e^{3-x-\frac{y}{2}}\right) \cdot \frac{3}{2}\, dx dy$$

$$= \int_0^4 \left[\frac{x^2}{2}y - e^{3-x-\frac{y}{2}} \right]_{x=0}^{x=1-\frac{y}{4}} dy$$

$$= \int_0^4 \left(\frac{y}{2}\left(1-\frac{y}{4}\right)^2 + 1 - e^{2-\frac{y}{4}} \right) dy$$

$$= \frac{1}{6} + 4 + 4(e - e^2).$$

$\square$