Note 17

Section 17.4 Parametrized Surfaces and Surface Integrals

② Studying surfaces using parametrizations

Standing Assumption $G : D \to \mathbb{R}$ is:

(1) one-to-one (i.e., does not parametrize the same point several times.)

(2) continuously differentiable (i.e., $x(u,v)$, $y(u,v)$, $z(u,v)$ have continuous partial derivatives).

DEF Let S be a surface parametrized by $G : D \to \mathbb{R}^3$.

(1) Images of horizontal & vertical lines in D under G are called grid curves on the surface.

(2) At a point $P = G(u_0, v_0)$ on S,

$$\mathbb{T}_u(P) = \frac{\partial G}{\partial u}(u_0, v_0)$$

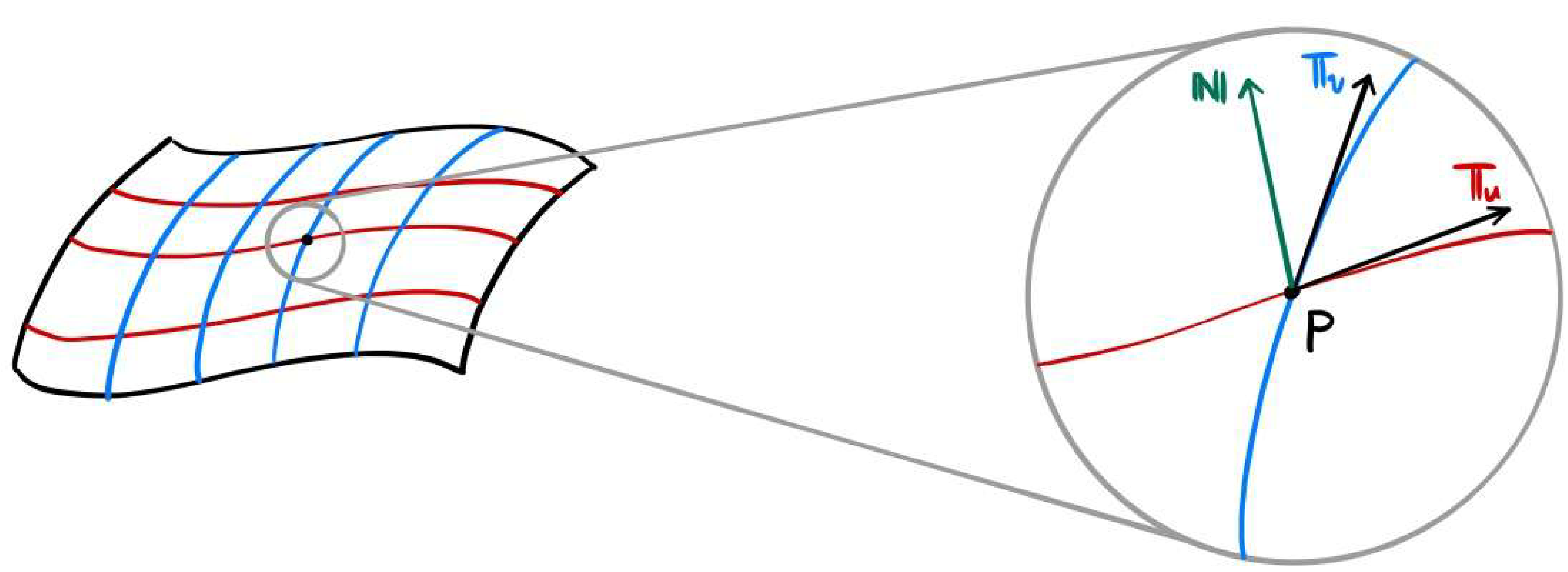
$$\mathbb{T}_v(P) = \frac{\partial G}{\partial v}(u_0, v_0)$$

are tangent vectors at P.

(3) The normal to the surface S is

$$\mathbb{N}(P) = \mathbb{T}_u(P) \times \mathbb{T}_v(P).$$

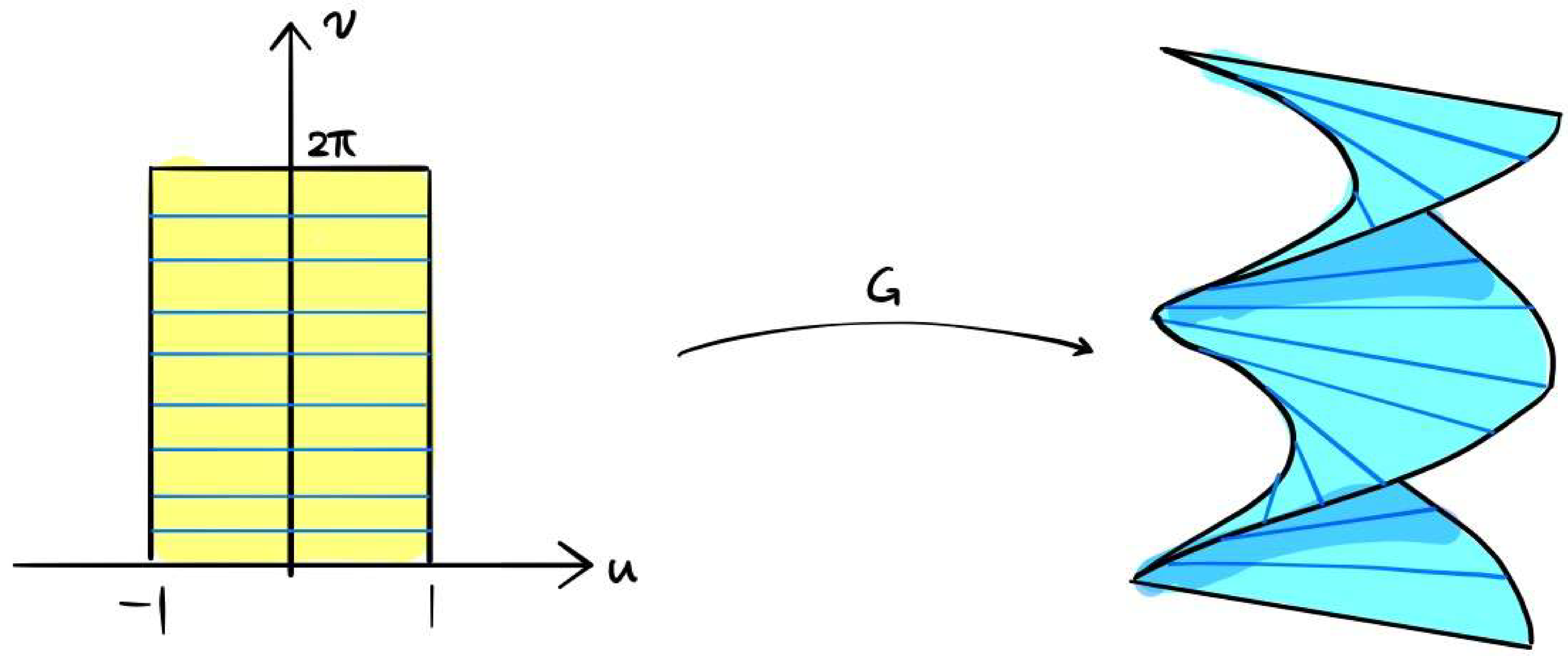
G is called regular if $\mathbb{N}(P) \neq 0$ for any points P on S.



Convention Although $\mathbb{T}_u$, $\mathbb{T}_v$, $\mathbb{N}$ are functions of points on S, we will often write $\mathbb{T}_u(u,v)$, $\mathbb{T}_v(u,v)$, $\mathbb{N}(u,v)$ to denote these vectors at point $G(u,v)$, for convenience.

Ex (Helicoid Surface) Let S be a surface with parametrization

$$G(u,v) = (u\cos v,\ u\sin v,\ v), \qquad -1 \le u \le u, \qquad 0 \le v \le 2\pi.$$



Then

$$\mathbb{T}_u = \langle \cos v,\ \sin v,\ 0 \rangle,$$
$$\mathbb{T}_v = \langle -u\sin v,\ u\cos v,\ 1 \rangle,$$

and so,

$$\mathbb{N} = \begin{vmatrix} \mathbb{i} & \mathbb{j} & \mathbb{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = (\sin v)\,\mathbb{i} - (\cos v)\,\mathbb{j} + u\,\mathbb{k}.$$

$\square$

Ex • If S : cylinder $G(\theta, z) = (R\cos\theta,\ R\sin\theta,\ z)$, then

$$\mathbb{N}(\theta, z) = \langle R\cos\theta,\ R\sin\theta,\ 0 \rangle$$

- If S : sphere $G(\theta,\phi) = (R\sin\phi\cos\theta,\ R\sin\phi\sin\theta,\ R\cos\phi)$, then

$$\mathbb{N}(\theta,\phi) = (R^2\sin\phi)\,\mathbb{e}_r,$$

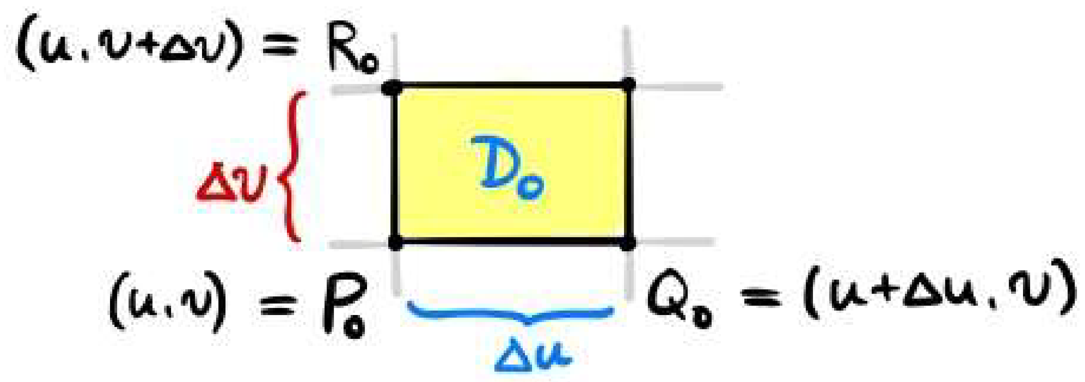
where

$$\mathbb{e}_r = \langle \sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi \rangle$$

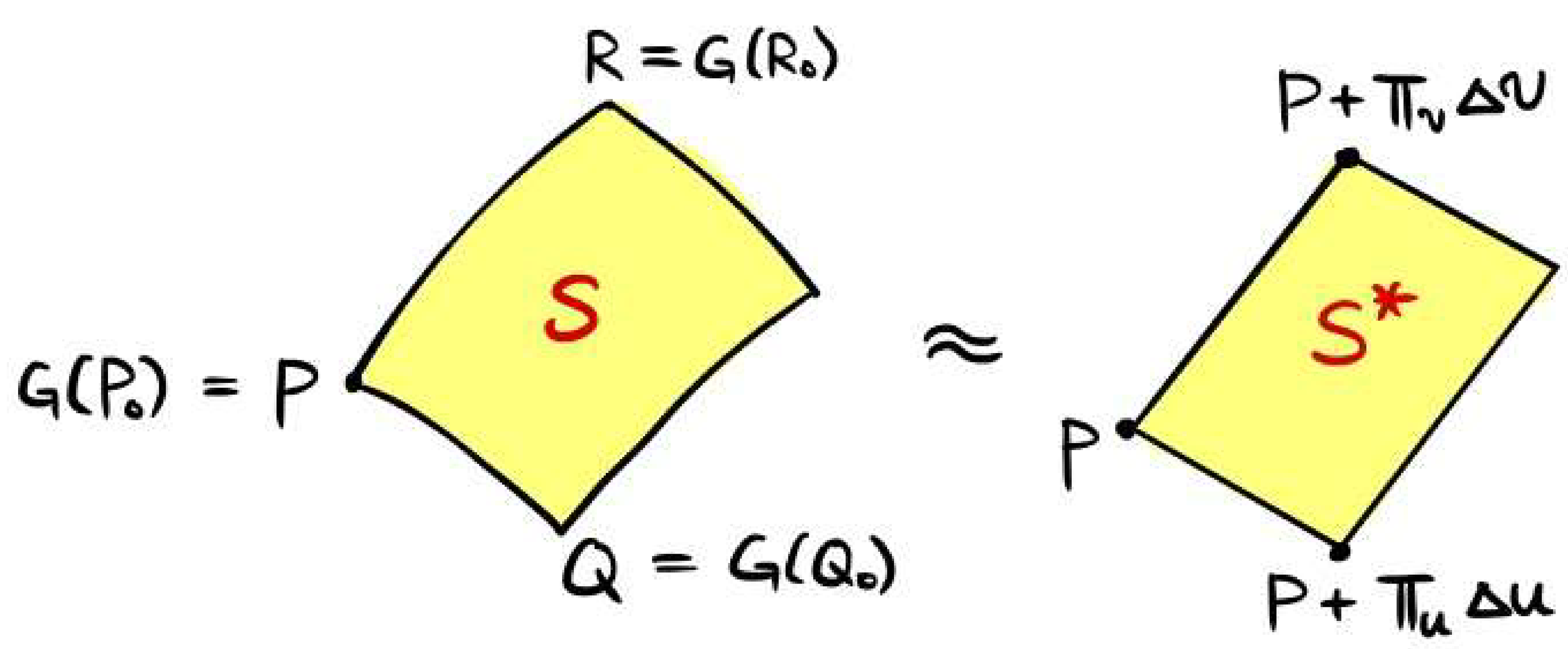
is the unit radial vector. $\square$

③ Surface Area

- If D_0 is a small rectangle in D as follows:



Then its image $S = G(D_0)$ is a "curved parallelogram":



Since

$$\overrightarrow{PQ} = G(u+\Delta u, v) - G(u,v) \approx \frac{\partial G}{\partial u}(u,v)\,\Delta u = \mathbb{T}_u\,\Delta u$$

and likewise

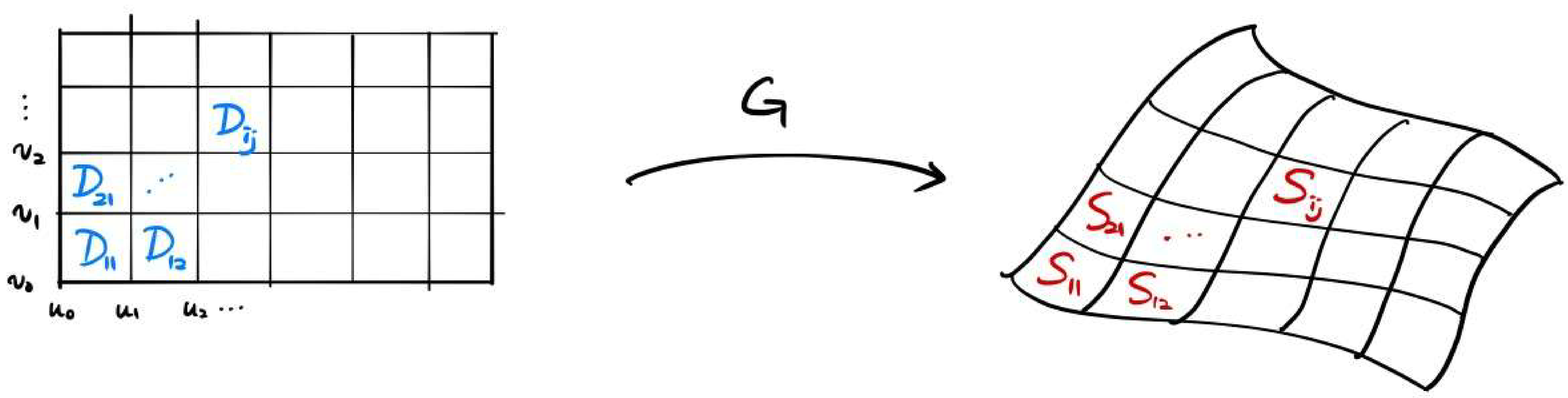
$$\overrightarrow{PR} \approx \mathbb{T}_v \Delta v,$$

The curved parallelogram S can be approximated by the parallelogram S^* spanned by $\mathbb{T}_u \Delta u$ and $\mathbb{T}_v \Delta v$, and so,

$$\begin{aligned} \text{area}(S) &\approx \text{area}(S^*) \\ &= \|(\mathbb{T}_u \Delta u) \times (\mathbb{T}_v \Delta v)\| \\ &= \|\mathbb{N}(u,v)\| \Delta u \Delta v \\ &= \|\mathbb{N}(u,v)\| \text{ area}(D_0). \end{aligned}$$

- Intermediate conclusion : $\|\mathbb{N}\|$ measures how the area of a small region scales under G.

- Now assume that G is regular and one-to-one. (Why?)
  Then splitting S into small patches S_{ij} arising from partitioning D into small rectangles,



we get

$$\text{area}(S) = \sum_{ij} \text{area}(S_{ij}) \approx \sum_{ij} \|\mathbb{N}(u_i, u_j)\| \Delta u \Delta v \approx \iint_D \|\mathbb{N}(u,v)\| \, du dv.$$

This becomes an identity as Δu and $\Delta v \to 0$:

DEF $$\text{area}(S) = \iint_D \|\mathbb{N}(u,v)\| \, du dv.$$

④ Surface Integral

- Borrowing the idea of Riemann sum, we may define the surface integral by :

DEF $$\iint_S f(x,y,z) \, dS := \lim_{\text{partition} \to 0} \sum_{ij} f(P_{ij}) \text{ area}(S_{ij})$$

- Then, by using the partition & sample points arising from those under parametrization, we get

$$\sum_{ij} f(P_{ij})\,\text{area}(S_{ij}) \approx \sum_{ij} f(G(u_i, v_j))\,\|\mathbb{N}(u_i, v_j)\|\,\Delta u \Delta v,$$

and so, passing to the limit gives:

THM Let $G : D \to S$ be a parametrization of the surface S such that

- G is continuously differentiable,
- G is one-to-one and regular (except possibly at the boundary of D).

Then

$$\iint_S f(x, y, z)\, dS = \iint_D f(G(u, v))\,\|\mathbb{N}(u, v)\|\, du\,dv.$$

- Remark) This theorem allows to write the surface-area differential dS as:

$$dS = \|\mathbb{N}(u, v)\|\, du\,dv.$$

Ex 1 Let $G(x, y) = (x, y, xy)$ over $D = \{(x, y) : x^2 + y^2 \le 1,\ \ x \geq 0,\ y \geq 0\}$. Then

$$\mathbb{T}_x = \frac{\partial G}{\partial x} = (1, 0, y),$$
$$\mathbb{T}_y = \frac{\partial G}{\partial y} = (0, 1, x),$$
$$\mathbb{N}(x, y) = \mathbb{T}_x \times \mathbb{T}_y = (-y, -x, 1).$$

So, if $S = G(D)$, then

$$\text{area}(S) = \iint_S 1\, dS = \iint_D \|\mathbb{N}(x, y)\|\, dx\,dy = \iint_D \sqrt{1 + x^2 + y^2}\, dx\,dy.$$

This may be computed using polar coordinates:

$$= \int_0^{\frac{\pi}{2}} \int_0^1 \sqrt{1 + r^2} \cdot r\,dr\,d\theta = \frac{\pi}{2} \cdot \left[\frac{1}{3}(1 + r^2)^{3/2} \right]_{r=0}^{r=1} = \frac{\pi}{6}(2\sqrt{2} - 1).$$

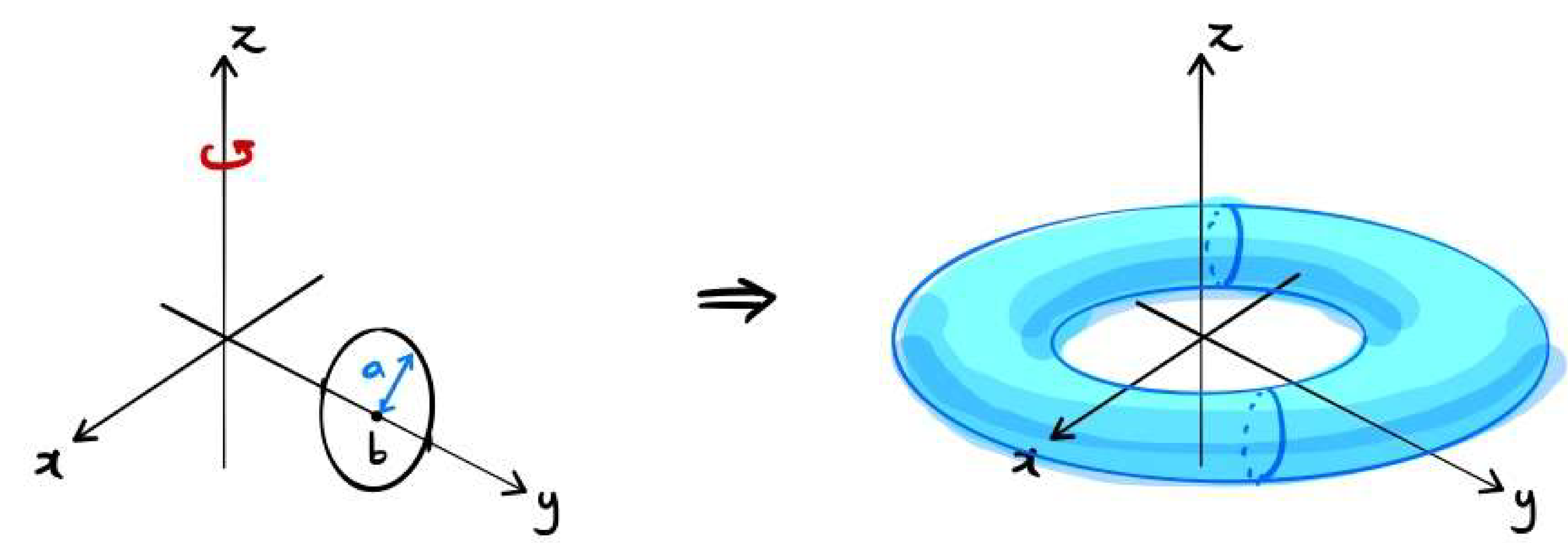
Similarly, the integral of $f(x,y,z) = z$ over S is

$$\iint_S z\, dS = \iint_D xy\sqrt{1+x^2+y^2} = \int_0^{\frac{\pi}{2}}\int_0^1 r^3\cos\theta\sin\theta\sqrt{1+r^2}\, dr\, d\theta$$

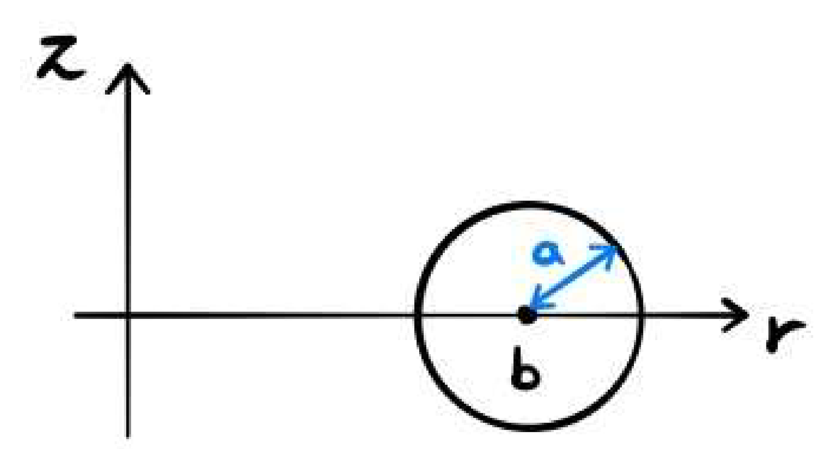
$$= \left[\frac{1}{2}\sin^2\theta\right]_0^{\frac{\pi}{2}} \cdot \left[\frac{(3r^2-2)(r^2+1)^{3/2}}{15}\right]_0^1 = \frac{1+\sqrt{2}}{15}.$$

$\square$

Ex 2 Compute the area of the torus S obtained by rotating the circle in the yz-plane $(y-b)^2 + z^2 = a^2$ about the z-axis. $(b > a > 0)$



Sol) Step 1. Parametrize S : Using the cylindrical coordinates,



$$\begin{cases} r = b + a\cos t \\ z = a\sin t \end{cases} \qquad (0 \le t \le 2\pi)$$

In terms of rectangular coordinates, $(x, y, z) = G(\theta, t)$ is given by

$$\begin{cases} x = r\cos\theta = (b + a\cos t)\cos\theta \\ y = r\sin\theta = (b + a\cos t)\sin\theta \\ z = a\sin t. \end{cases} \qquad 0 \le \theta \le 2\pi, \quad 0 \le t \le 2\pi.$$

Step 2. Compute $\mathbb{T}_\theta$, $\mathbb{T}_t$, $\mathbb{N}(\theta, t)$.

$$\mathbb{T}_\theta = \frac{\partial G}{\partial \theta} = (b + a\cos t)\langle -\sin t, \cos t, 0\rangle.$$

$$\mathbb{T}_t = \frac{\partial G}{\partial t} = a\langle -\sin t\cos\theta, -\sin t\sin\theta, \cos t\rangle.$$

$$\mathbb{N}(\theta, t) = \mathbb{T}_\theta \times \mathbb{T}_t = a(b + a\cos t)\langle \cos t\cos\theta,\ \cos t\sin\theta,\ \sin t\rangle.$$

Step 3. Calculate the surface area:

$$\begin{aligned}
\text{area}(S) = \iint_S 1\, dS &= \int_0^{2\pi}\int_0^{2\pi} \|\mathbb{N}(\theta, t)\|\, d\theta\, dt \\
&= \int_0^{2\pi}\int_0^{2\pi} a(a + b\cos t)\, d\theta\, dt \\
&= 4\pi^2 ab.
\end{aligned}$$

$\square$