Note 16

Section 17.3 Conservative Vector Field

Recall) • We learned :

$$(\mathbb{F} \text{ is path-indep.}) \iff (\mathbb{F} \text{ is conservative})$$

$$\begin{array}{c} (\mathbb{F} \text{ is conservative}) \\ \Downarrow \quad \Uparrow \ \text{if the domain is simply connected} \\ (\mathbb{F} \text{ satisfies CPC}) \end{array}$$

Ex (vortex field revisited) Let $\mathbb{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ be the vortex field on the upper half-plane

$$D = \{(x,y) : y > 0\}.$$

Then

$$f(x,y) = \frac{\pi}{2} - \arctan(x/y)$$

satisfies $\nabla f(x,y) = \mathbb{F}(x,y)$, i.e., f is a potential function of $\mathbb{F}$ on D. That $\mathbb{F}$ is conservative on D does not contradict the fact that $\mathbb{F}$ is not conservative on all of $\mathbb{R}^2$ minus 0, since the "conservative-ness" depends on the choice of the domain. □

Ex (Finding a potential function) Show that

$$\mathbb{F}(x,y) = \langle 2xy^3 + e^x, \ 3x^2y^2 - \sin(y) \rangle$$

is conservative and find all the potential functions of $\mathbb{F}$.

Sol) ① Since the domain of $\mathbb{F}$ is all of $\mathbb{R}^2$, which is simply-connected,

(1) $\mathbb{F}$ is conservative $\iff$ $\mathbb{F}$ satisfies the cross-partial condition, and

(2) any two potential functions of $\mathbb{F}$ differ only by constant.

② Cross-Partial condition?

$$\frac{\partial}{\partial y}\left(2xy^3 + e^x\right) = 6xy^2,$$

$$\frac{\partial}{\partial x}\left(3x^2y^2 - \sin y\right) = 6x^2y^2.$$

So $\mathbb{F}$ satisfies CPC $\Rightarrow$ $\mathbb{F}$ is conservative.

③ Let f be a potential function of $\mathbb{F}$. Then

$$\frac{\partial f}{\partial x} = 2xy^3 + e^x$$

★ constant of integration in dx may still depend on y

$$\Rightarrow f(x,y) = \int \left(2xy^3 + e^x\right) dx = x^2y^3 + e^x + \boxed{g(y)}$$

Similarly,

$$\frac{\partial f}{\partial y} = 3x^2y^2 - \sin y$$

$$\Rightarrow f(x,y) = \int \left(3x^2y^2 - \sin y\right) dy = x^2y^3 + \cos y + \boxed{h(x)}$$

Since two answers must coincide, we may set

$$g(y) = \cos y + C, \qquad h(x) = e^x + C$$

for a constant C, and hence

$$f(x,y) = x^2y^3 + e^x + \cos y + C.$$

□

Ex Find a potential function for

$$\mathbb{F}(x,y,z) = \left\langle \frac{y}{z},\ z+\frac{x}{z},\ y-\frac{xy}{z^2}+1 \right\rangle$$

Sol) Let f be a potential ftn. Then

(1) $f(x,y,z) = \int F_1(x,y,z)\,dx = \frac{xy}{z} + g(y,z)$

(2) $f(x,y,z) = \int F_2(x,y,z)\,dy = yz + \frac{xy}{z} + h(x,z)$

(3) $f(x,y,z) = \int F_3(x,y,z)\,dz = yz + \frac{xy}{z} + z + k(x,y).$

Since these must hold simultaneously, we have:

$$f(x,y,z) = yz + \frac{xy}{z} + z + C.$$

$\square$

Section 17.4 Parametrized Surface and Surface Integral

① Parametrized Surfaces

DEF • A **parametrized surface** is the image of $D \subseteq \mathbb{R}^2$ under the map

$$G : D \to \mathbb{R}^3$$

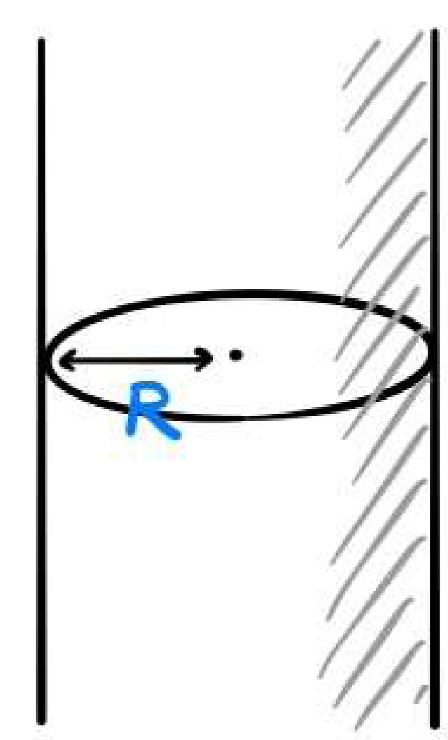
where

$$G(u, v) = (x(u,v),\ y(u,v),\ z(u,v)).$$

• In this context, u & v are called parameters.

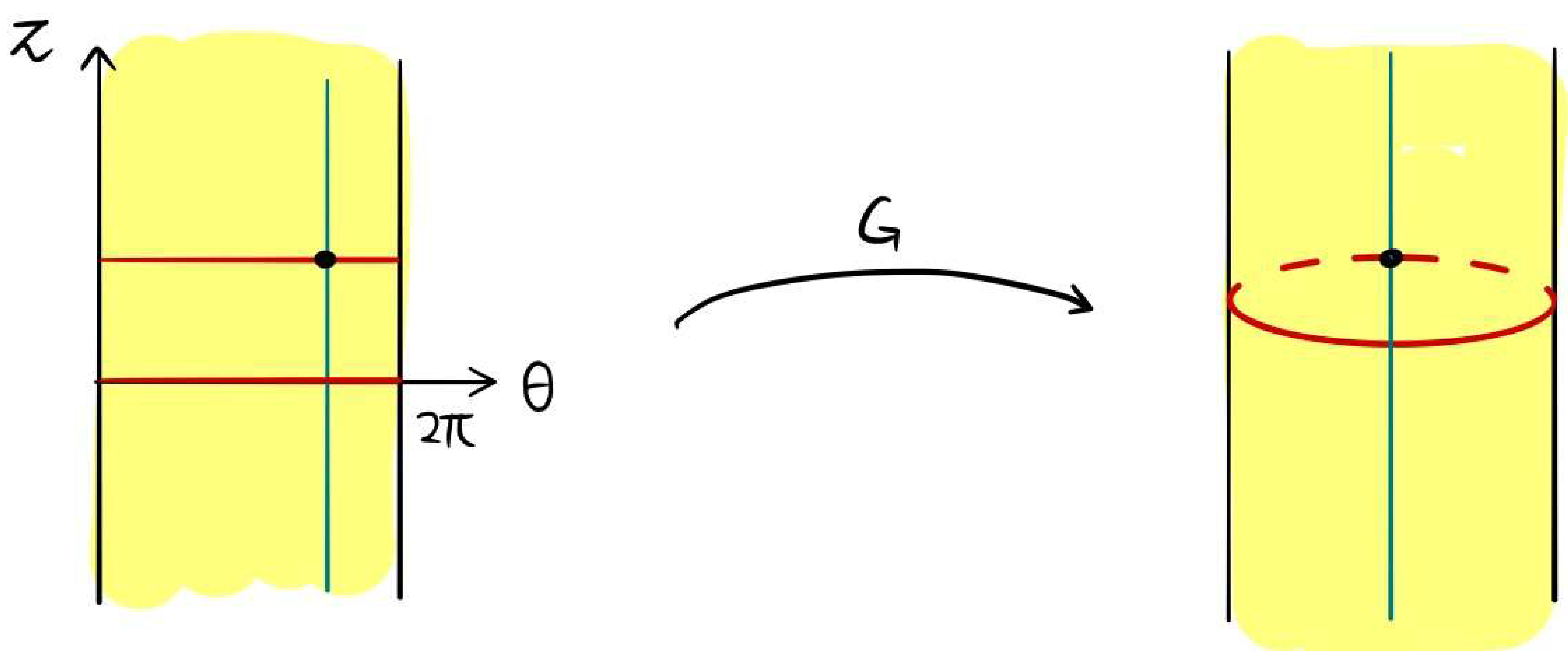
Example 1 (Cylinder) Points on the cylinder

$$x^2 + y^2 = R^2$$



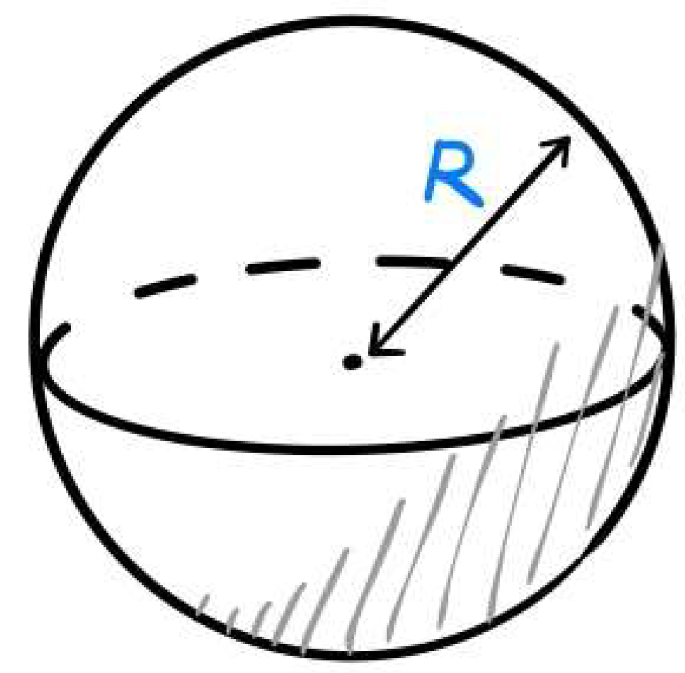
have cylindrical coordinates (R, θ, z), and so, we may use θ and z as parameters to give:

$$G(\theta, z) = (R\cos\theta,\ R\sin\theta,\ z), \qquad 0 \leq \theta \leq 2\pi, \quad z \in \mathbb{R}.$$



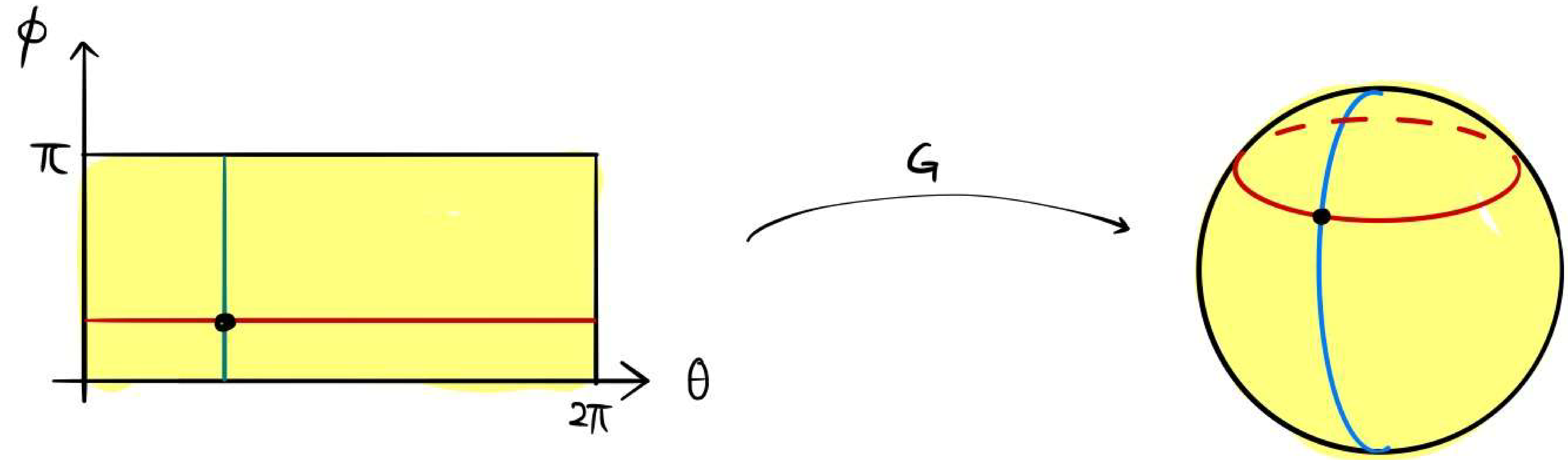
Example 2 (Sphere) Points on the sphere

$$x^2+y^2+z^2=R^2$$



have spherical coordinates (R, θ, ϕ), and so, we may use θ and ϕ as parameters to give:

$$G(\theta, \phi) = (R\sin\phi\cos\theta,\ R\sin\phi\sin\theta,\ R\cos\phi), \qquad 0 \le \theta \le 2\pi,\ \ 0 \le \phi \le \pi.$$

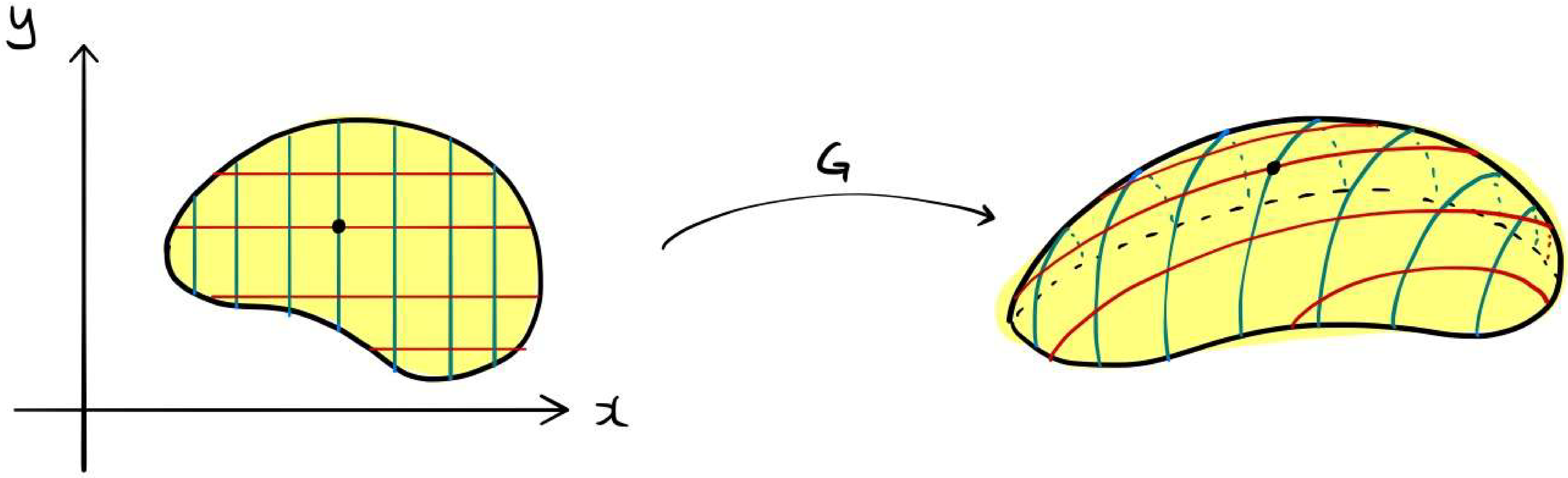


Example 3 (Graph) Points on the graph

$$z = f(x,y) \quad \text{over } D$$

can be parametrized by

$$G(x,y) = (x,\ y,\ f(x,y)), \qquad (x,y) \in D.$$



② Studying surfaces using parametrizations

Standing Assumption $G : D \to \mathbb{R}$ is :

(1) one-to-one (i.e., does not parametrize the same point several times.)

(2) continuously differentiable (i.e., $x(u,v)$, $y(u,v)$, $z(u,v)$ have continuous partial derivatives).

DEF Let S be a surface parametrized by $G : D \to \mathbb{R}^3$.

(1) Images of horizontal & vertical lines in D under G are called grid curves on the surface.

(2) At a point $P = G(u_0, v_0)$ on S,

$$\mathbb{T}_u(P) = \frac{\partial G}{\partial u}(u_0, v_0)$$

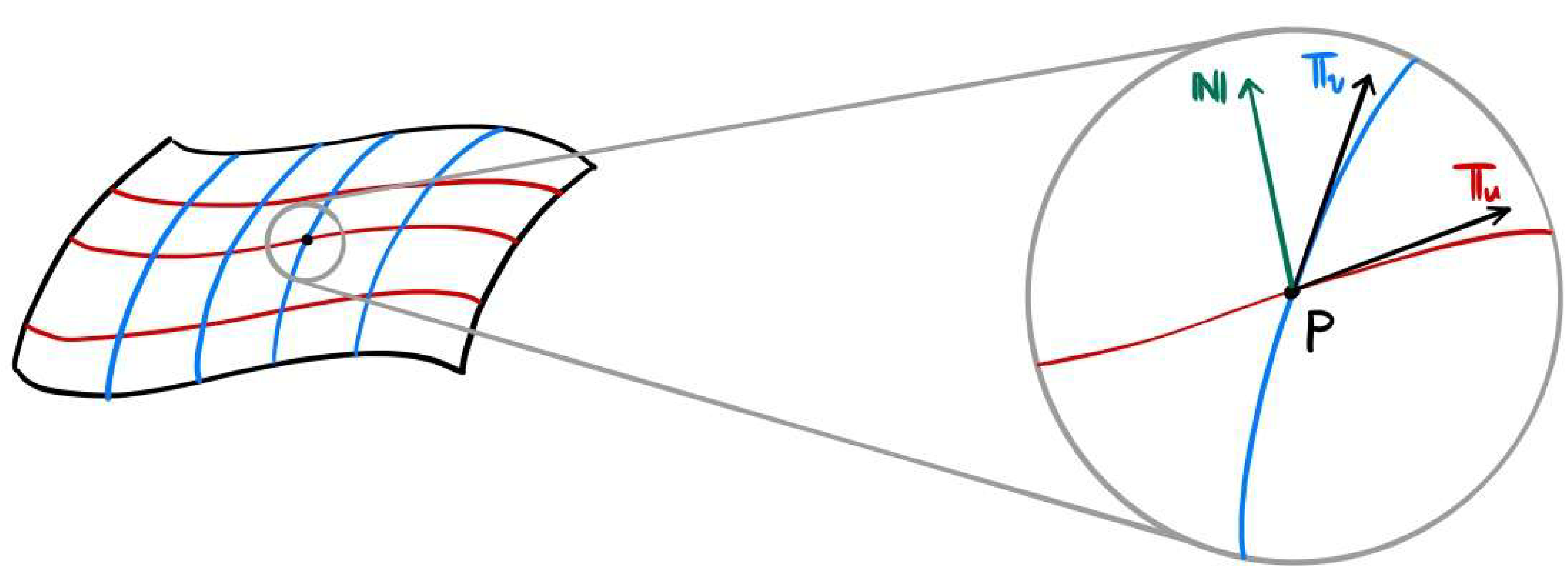
$$\mathbb{T}_v(P) = \frac{\partial G}{\partial v}(u_0, v_0)$$

are tangent vectors at P.

(3) The normal to the surface S is

$$\mathbb{N}(P) = \mathbb{T}_u(P) \times \mathbb{T}_v(P).$$

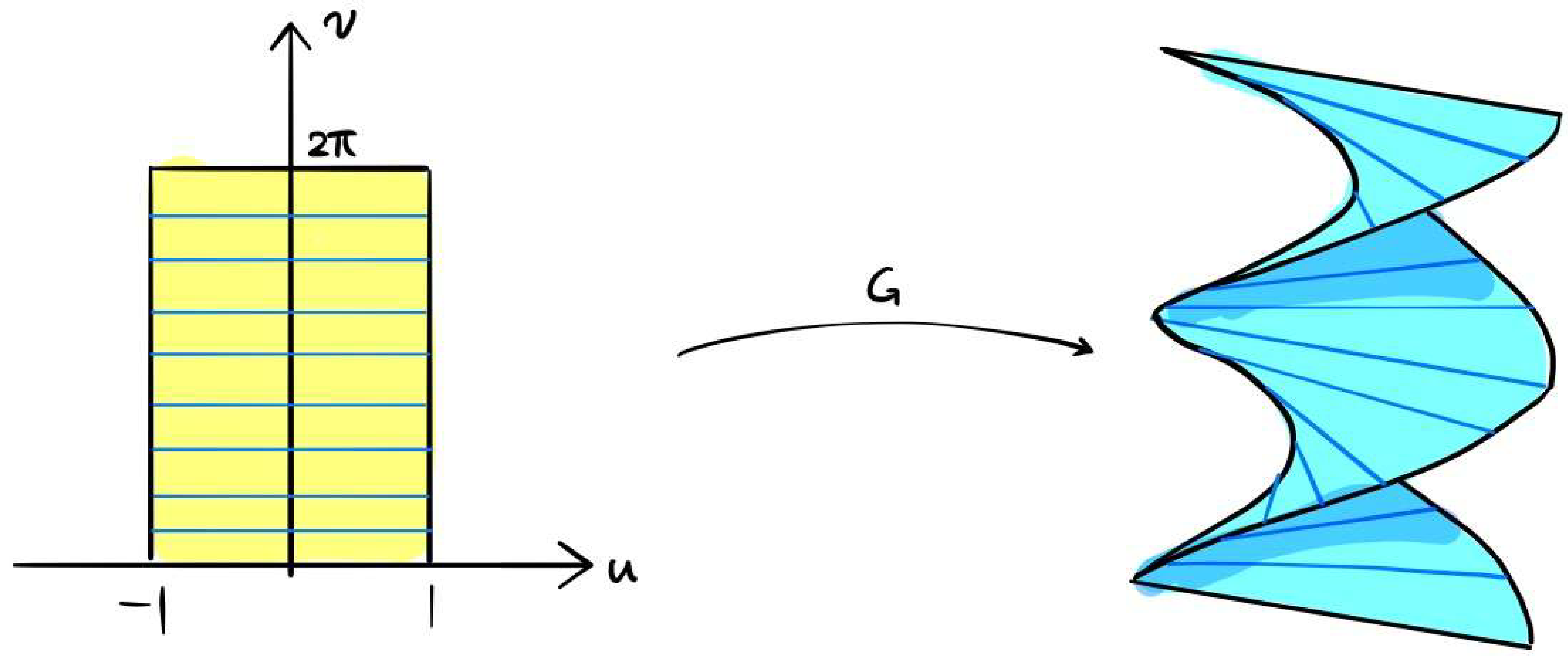
G is called regular if $\mathbb{N}(P) \neq 0$ for any points P on S.



Convention Although $\mathbb{T}_u$, $\mathbb{T}_v$, $\mathbb{N}$ are functions of points on S, we will often write $\mathbb{T}_u(u,v)$, $\mathbb{T}_v(u,v)$, $\mathbb{N}(u,v)$ to denote these vectors at point $G(u,v)$, for convenience.

Ex (Helicoid Surface) Let S be a surface with parametrization

$$G(u,v) = (u\cos v,\ u\sin v,\ v), \qquad -1 \le u \le u, \qquad 0 \le v \le 2\pi.$$



Then

$$\mathbb{T}_u = \langle \cos v,\ \sin v,\ 0 \rangle,$$
$$\mathbb{T}_v = \langle -u\sin v,\ u\cos v,\ 1 \rangle,$$

and so,

$$\mathbb{N} = \begin{vmatrix} \mathbb{i} & \mathbb{j} & \mathbb{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = (\sin v)\,\mathbb{i} - (\cos v)\,\mathbb{j} + u\,\mathbb{k}.$$

□

Ex • If S : cylinder $G(\theta, z) = (R\cos\theta,\ R\sin\theta,\ z)$, then

$$\mathbb{N}(\theta, z) = \langle R\cos\theta,\ R\sin\theta,\ 0 \rangle$$

- If S : sphere $G(\theta, \phi) = (R\sin\phi\cos\theta,\ R\sin\phi\sin\theta,\ R\cos\phi)$, then

$$\mathbb{N}(\theta, \phi) = (R^2 \sin\phi)\, \mathbb{e}_r,$$

where

$$\mathbb{e}_r = \langle \sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi \rangle$$

is the unit radial vector.