Note 14

Section 17.2.

☒ Application

① [total amount over C] $= \int_C$ [density function] ds.

Ex Total charge on the curve $y = x^{4/3}$ for $1 \le x \le 8$, assuming the charge density of $\rho(x,y) = x/y$:

$$\int_C \frac{x}{y}\, ds = \int_1^8 \frac{x}{x^{4/3}} \sqrt{1 + \left(\frac{4}{3}x^{1/3}\right)^2}\, dx$$

$$= \int_1^8 \frac{1}{x^{1/3}} \sqrt{1 + \frac{16}{9}x^{2/3}}\, dx$$

$$\left(u = 1 + \frac{16}{9}x^{2/3}\right) = \int_{\frac{25}{9}}^{\frac{73}{9}} \frac{27}{32}\sqrt{u}\, du$$

$$= \left[\frac{9}{16}u^{\frac{3}{2}}\right]_{\frac{25}{9}}^{\frac{73}{9}} = \frac{1}{48}\left(73^{\frac{3}{2}} - 125\right).$$

② Electric Potential

- Coulomb's law tells that

$$\underbrace{\text{[electric potential at } P]}_{V(P)} = \int_C \overset{\text{some physics constant}}{k} \frac{\text{[charge density function]}}{\text{[dist. from } P \text{ to } (x,y,z)]}\, ds$$

Ex Let C : line segment from $(0,1,0)$ to $(1,1,0)$ in meters,
$\rho(x,y,z) = 10^{-8} \cdot x$

Then

$$V(0,0,a) = \int_0^1 \frac{10^{-8}\cdot x}{\sqrt{x^2+1^2+a^2}}\,dx$$

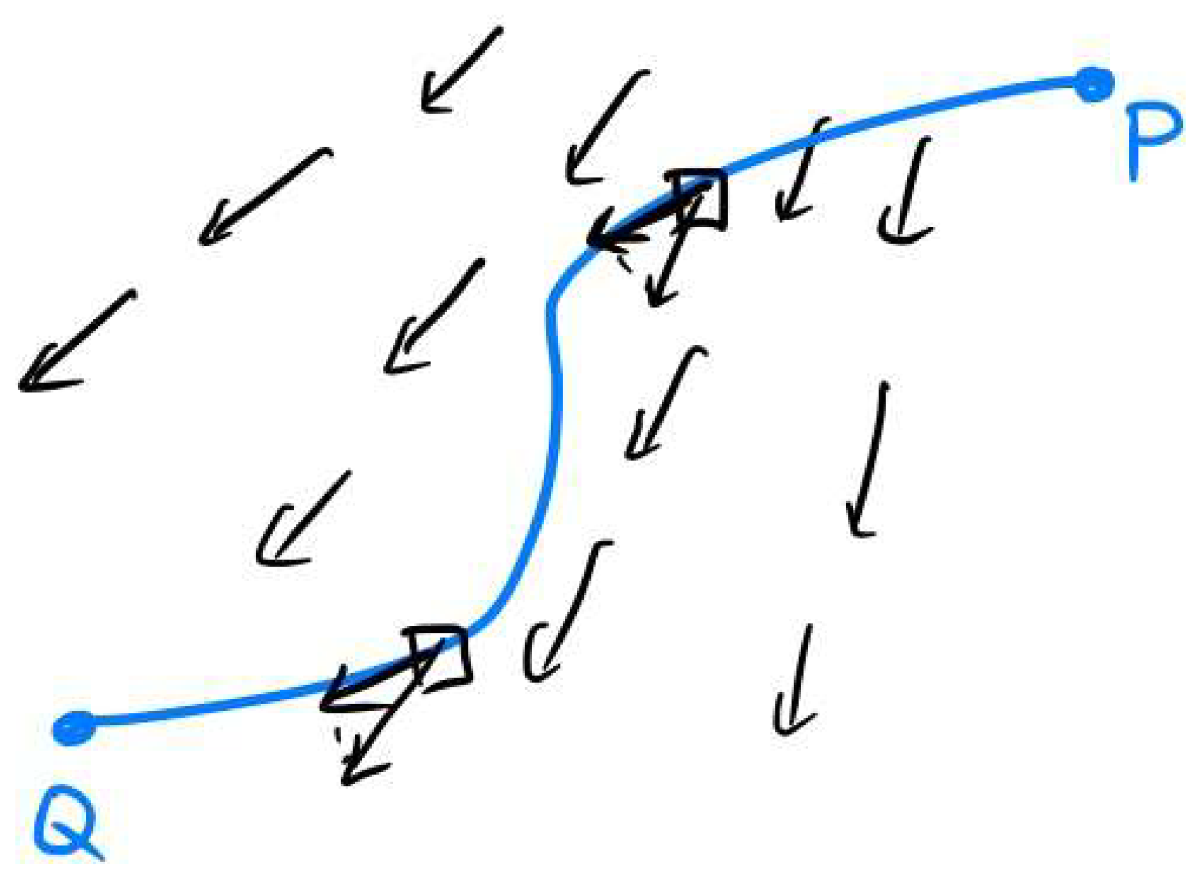
$$= \left[10^{-8}\sqrt{x^2+1+a^2}\right]_0^1 = 10^{-8}\left(\sqrt{2+a^2}-\sqrt{1+a^2}\right).$$

③ Work.

- In physics,

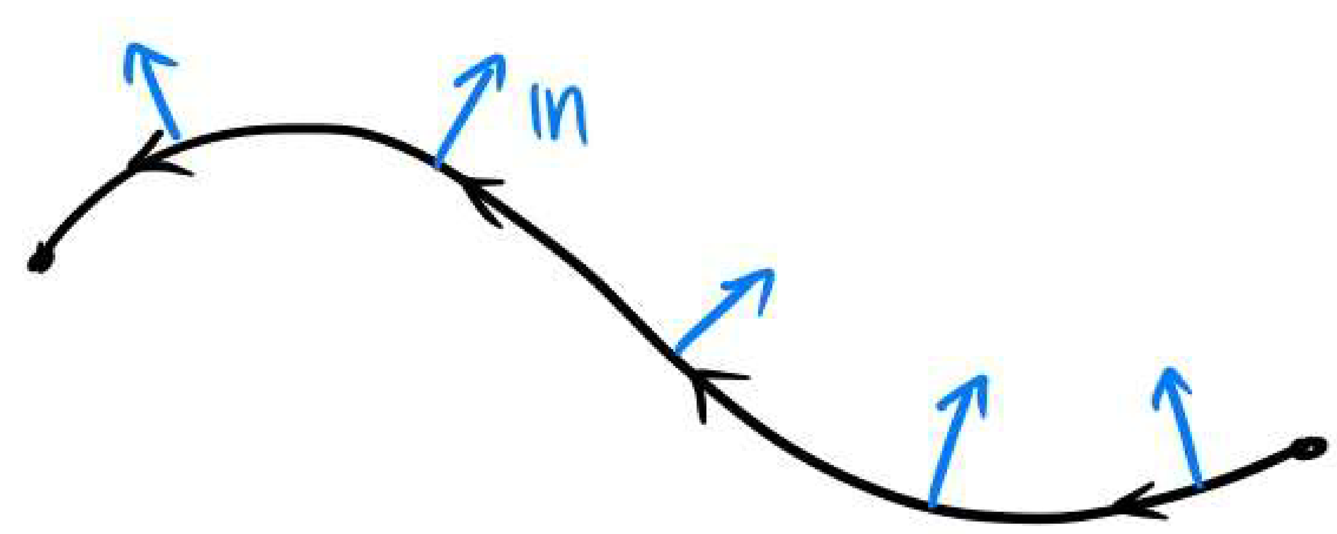
  [work performed by $\mathbb{F}$] $= \int_C \mathbb{F}\cdot d\mathbb{r}$.

  [work performed against $\mathbb{F}$] $= -\int_C \mathbb{F}\cdot d\mathbb{r}$.



④ Flux in 2D:

- If C : oriented curve, the unit normal vector $\mathbb{n}$ is



the unit vector pointing the direction going from left to right.

- The flux of $\mathbb{F}$ across $\mathcal{C}$ is defined by

$$\int_{\mathcal{C}} (\mathbb{F} \cdot \mathbb{n})\, ds.$$

THM We have

$$\int_{\mathcal{C}} (\mathbb{F} \cdot \mathbb{n})\, ds = \int_{\mathcal{C}} F_1\, dy - F_2\, dx.$$

Pf) ▷ Let $\mathbb{r}(t)$: parametrization of $\mathcal{C}$.

▷ $\mathbb{n}$ is the -90° rotation of $\mathbb{T}$, and so,

$$\begin{aligned} \mathbb{n}(\mathbb{r}(t))\, \|\mathbb{r}'(t)\| &= (-90^\circ) \text{ rotation of } \mathbb{r}'(t) \\ &= \langle y'(t), -x'(t) \rangle, \end{aligned}$$

and so,

$$\begin{aligned} \int_{\mathcal{C}} (\mathbb{F} \cdot \mathbb{n})\, ds &= \int_a^b \mathbb{F}(\mathbb{r}(t)) \cdot \mathbb{n}(\mathbb{r}(t))\, \|\mathbb{r}'(t)\|\, dt \\ &= \int_a^b \langle F_1, F_2 \rangle \cdot \langle y'(t), -x'(t) \rangle\, dt \\ &= \int_a^b \left(F_1\, y'(t) - F_2\, x'(t) \right) dt \\ &= \int_{\mathcal{C}} F_1\, dy - F_2\, dx. \end{aligned}$$

$\square$

Ex Compute the flux of $\mathbb{F} = \left\langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right\rangle$ across the circle $x^2+y^2 = R^2$ oriented CCW.

Sol)

$$\begin{aligned} [\text{flux}] &= \int_{\mathcal{C}} \frac{x\,dy - y\,dx}{x^2+y^2} = \int_0^{2\pi} \frac{R^2\cos^2\theta - R^2\sin\theta(-\sin\theta)}{R^2}\, d\theta \\ &= 2\pi. \end{aligned}$$

$\square$