Note 13

Section 17.2. Line Integrals

Last time: Defined scalar line integral using the idea of Riemann sums

Today:
- Investigate some applications in physics.
- Learn vector line integral

2 Applications of Scalar Line Integral

- If $\rho(x,y,z)$ denotes density over $\mathcal{C}$, then

$$[\text{total amount over } \mathcal{C}] = \int_{\mathcal{C}} \rho(x,y,z)\, ds$$

- If $\rho(x,y,z)$ denotes charge density over $\mathcal{C}$, then Coulomb's law tells:

$$[\text{electric potential at } P] = k \int_{\mathcal{C}} \frac{\rho(x,y,z)}{\underbrace{\|\langle x,y,z\rangle - P\|}_{\text{distance from } \langle x,y,z\rangle \text{ to } P.}}\, ds$$

some physical constant

3 Vector Line Integral

① Definition

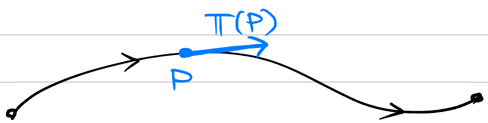
"traversing direction"

- Setting: Suppose we have
  - ▷ an oriented curve $\mathcal{C}$ (curve w/ orientation)
  - ▷ a vector field $\mathbb{F}$ over $\mathcal{C}$.

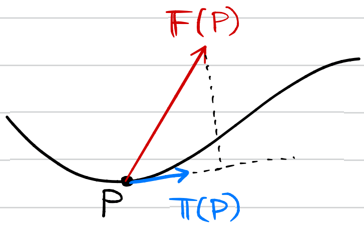
(Same as mere curves, different as oriented curves.)

- If $\mathcal{C}$: oriented curve & P : point on $\mathcal{C}$, we write

$$\mathbb{T}(P) = [\text{unit tangent vector at } P]$$



- $\mathbb{F} \cdot \mathbb{T}$ is the tangential component of $\mathbb{F}$ at P :



DEF The vector line integral of $\mathbb{F}$ over $\mathcal{C}$ is defined as

$$\int_{\mathcal{C}} (\mathbb{F} \cdot \mathbb{T})\, ds$$

and is denoted by

$$\int_{\mathcal{C}} \mathbb{F} \cdot d\mathbb{r} \quad \text{or} \quad \int_{\mathcal{C}} (F_1\, dx + F_2\, dy + F_3\, dz)$$

- $d\mathbb{r} = \langle dx, dy, dz \rangle = \mathbb{T}\, ds$ is often called vector differential.

Q Is this strange thing ever useful?

A Absolutely! ☺

② Parametrization

- If $\mathbf{r}$: parametrization of $\mathcal{C}$, then its orientation (traverse direction) may be either :

positive (the same as the orientation of $\mathcal{C}$) or negative (the opposite of the orientation of $\mathcal{C}$)

- In scalar line integral, both can be used. BUT, since

$$\mathbb{T} = \begin{cases} \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} & \text{for positively oriented} \\ -\dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} & \text{for negatively oriented,} \end{cases}$$

we have to distinguish both cases in vector line integral.

THM If $\mathbf{r} : [a,b] \to \mathcal{C}$ is positively oriented param., then

$$\int_{\mathcal{C}} \mathbb{F} \cdot d\mathbf{r} = \int_a^b \mathbb{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.$$

- If we write $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, then

$$\int_{\mathcal{C}} \mathbb{F} \cdot d\mathbf{r} = \int_a^b \left(F_1(\mathbf{r}(t)) \cdot \frac{dx(t)}{dt} + \cdots + F_3(\mathbf{r}(t)) \frac{dz(t)}{dt} \right) dt.$$

Ex Evaluate $\int_{\mathcal{C}} \mathbb{F} \cdot d\mathbb{r}$, where

$\mathbb{F} = \langle y, e^z, x+z \rangle$,

$\mathcal{C}$: param-ed by $\mathbb{r}(t) = \langle t+1, t^2, 2 \rangle \quad (1 \le t \le 2)$

Sol) $\mathbb{F}(\mathbb{r}(t)) \cdot \mathbb{r}'(t) = \langle t^2, e^2, t+3 \rangle \cdot \langle 1, 2t, 0 \rangle$
$= t^2 + 2e^2 t + (t+3)$

So,

$$\int_{\mathcal{C}} \mathbb{F} \cdot d\mathbb{r} = \int_1^2 (t^2 + (2e^2+1)t + 3)\, dt$$
$$= \left[\frac{t^3}{3} + (2e^2+1) \cdot \frac{t^2}{2} + 3t \right]_1^2 = \cdots \qquad \square$$

Ex Let

- $\mathcal{C}$: circle of radius R at O, oriented CCW.
- $\mathbb{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$.

Then $\mathcal{C}$ may be param-ed by

$\mathbb{r}(t) = \langle R\cos t, R\sin t \rangle$. $\quad (0 \le t \le 2\pi)$

$\Rightarrow \mathbb{r}'(t) = \langle -R\sin t, R\cos t \rangle$

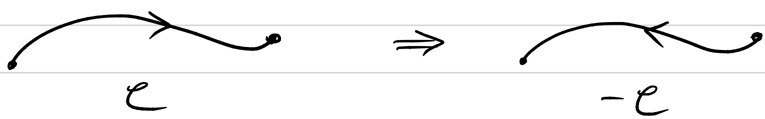
$\Rightarrow \int_{\mathcal{C}} \mathbb{F} \cdot d\mathbb{r}$

$$= \int_0^{2\pi} \left\langle \frac{-R\sin t}{R^2}, \frac{R\cos t}{R^2} \right\rangle \cdot \langle -R\sin t, R\cos t \rangle\, dt$$

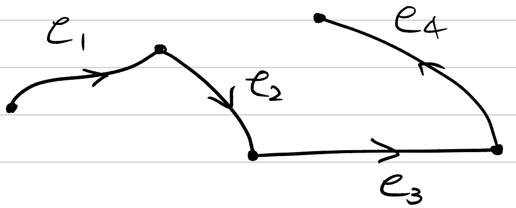
$$= \int_0^{2\pi} dt = 2\pi. \qquad \square$$

③ Extensions & Properties

- $-\mathcal{C}$: curve $\mathcal{C}$ with the opposite orientation.



- $\mathcal{C}_1 + \mathcal{C}_2 + \cdots + \mathcal{C}_n$: union of curves $\mathcal{C}_1, \cdots, \mathcal{C}_n$



$= \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4.$

PROP

- (Linearity)

$$\int_{\mathcal{C}} (a\mathbb{F} + b\mathbb{G}) \cdot d\mathbb{r} = a \int_{\mathcal{C}} \mathbb{F} \cdot d\mathbb{r} + b \int_{\mathcal{C}} \mathbb{G} \cdot d\mathbb{r}$$

- (Reversing orientation)

$$\int_{-\mathcal{C}} \mathbb{F} \cdot d\mathbb{r} = -\int_{\mathcal{C}} \mathbb{F} \cdot d\mathbb{r}$$

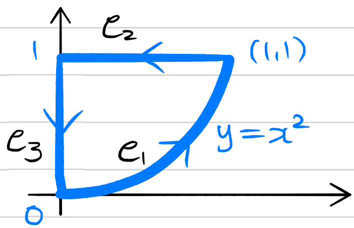
- (Additivity)

$$\int_{\mathcal{C}_1 + \cdots + \mathcal{C}_n} \mathbb{F} \cdot d\mathbb{r} = \int_{\mathcal{C}_1} \mathbb{F} \cdot d\mathbb{r} + \cdots + \int_{\mathcal{C}_n} \mathbb{F} \cdot d\mathbb{r}.$$

Ex Compute $\int_{\mathcal{C}} \mathbb{F} \cdot d\mathbb{r}$, where

$\mathbb{F} = \langle y, -x^2 \rangle$,

$\mathcal{C} =$



Sol) • Write $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3$ as above. Then

$\mathcal{C}_1 : \mathbb{r}(t) = \langle t, t^2 \rangle, \quad 0 \le t \le 1,$

$\mathcal{C}_2 : \mathbb{r}(t) = \langle 1-t, 1 \rangle, \quad 0 \le t \le 1.$

$\mathcal{C}_3 : \mathbb{r}(t) = \langle 0, 1-t \rangle, \quad 0 \le t \le 1.$

(CAUTION : $t \mapsto \langle t, 1 \rangle$ will traverse $-\mathcal{C}_2$, not $\mathcal{C}_2$).

Then

$$\int_{\mathcal{C}} \mathbb{F} \cdot d\mathbb{r} = \int_{\mathcal{C}} (y\,dx - x^2\,dy)$$

$$= \sum_{i=1}^{3} \int_{\mathcal{C}_i} (y\,dx - x^2\,dy),$$

and

$$\int_{\mathcal{C}_1} (y\,dx - x^2\,dy) = \int_0^1 \left(y \frac{dx}{dt} - x^2 \frac{dy}{dt} \right) dt$$

$$= \int_0^1 \left(t^2 \cdot 1 - t^2 \cdot 2t \right) dt = -\frac{1}{6}.$$

$$\int_{\mathcal{C}_2} (y\,dx - x^2\,dy) = \int_0^1 \left(1 \cdot (-1) - (1-t)^2 \cdot 0 \right) dt = -1$$

along $\mathcal{C}_2$, dy "=" 0.

$$\int_{\mathcal{C}_3} (y^2 \underbrace{dx}_{\text{"="}0 \text{ over } \mathcal{C}_3} - x^2 dy) = \int_0^1 \left((1-t)^2 \cdot 0 + 0 \cdot (-1) \right) dt = 0.$$

$$\therefore \int_{\mathcal{C}} \mathbb{F} \cdot d\mathbb{r} = -\frac{1}{6} - 1 + 0 = -\frac{7}{6}. \qquad \square$$