Note 12

Section 17.2. Line Integrals

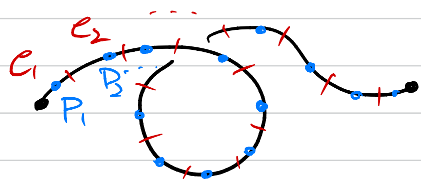
[I] Scalar Line Integrals

- Suppose we have:
  - ▷ a curve $\mathcal{C}$.
  - ▷ a (scalar) function f over $\mathcal{C}$.

<u>WANT</u> to find the "total amount" of f over $\mathcal{C}$.

① Definition

As usual, we use Riemann sums!



curve $\mathcal{C}$

- ▷ Partition $\mathcal{C}$ into N small pieces $\mathcal{C}_1, \cdots, \mathcal{C}_N$,
- ▷ Choose a sample point P_i on each $\mathcal{C}_i$,
- ▷ Write $\Delta s_i = \text{length}(\mathcal{C}_i)$.

Then, the scalar line integral of f over $\mathcal{C}$ is defined by

$$\int_{\mathcal{C}} f(x,y,z)\, ds := \lim_{\{\Delta s_i\} \to 0} \sum_{i=1}^{N} f(P_i)\, \Delta s_i .$$

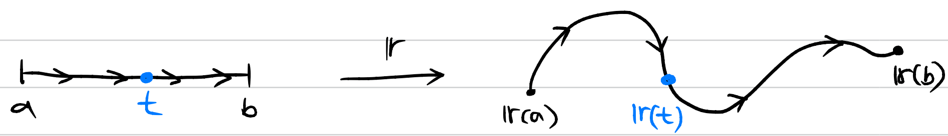
<u>Rmk)</u>

$$\int_{\mathcal{C}} 1\, ds = \text{length}(\mathcal{C}).$$

② Parametrizations

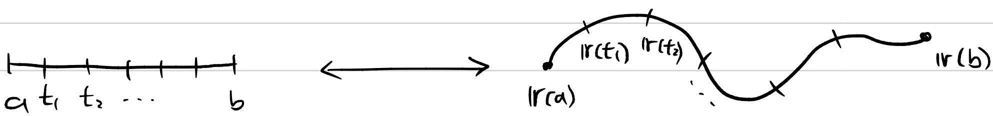
- In practice, the scalar line integrals are computed using parametrizations.

- Let $\mathbf{r}(t)$ over $a \le t \le b$ be a parametrization directly traversing C:



Then

$$\begin{pmatrix}\text{Partitioning } C \text{ into pieces} \\ C_1, \cdots, C_N\end{pmatrix} \longleftrightarrow \begin{pmatrix}\text{Partitioning } [a,b] \text{ into} \\ a = t_0 < t_1 < \cdots < t_N = b\end{pmatrix}$$



and

- ▷ $C_i \approx$ line segment joining $\mathbf{r}(t_{i-1})$ to $\mathbf{r}(t_i)$,
- ▷ $f(P_i) = f(\mathbf{r}(t_i^*))$ $\quad (t_i^* \in [t_{i-1}, t_i])$
- ▷ $\Delta S_i \approx$ length of $\overline{\mathbf{r}(t_{i-1})\mathbf{r}(t_i)} \approx \|\mathbf{r}'(t_i^*)\| \, \Delta t_i$.

So

$$\sum_{i=1}^{N} f(P_i)\, \Delta S_i \approx \sum_{i=1}^{N} f(\mathbf{r}(t_i^*))\, \|\mathbf{r}'(t_i^*)\|\, \Delta t_i$$

Taking limit as "mesh size" $\to 0$, we get:

THM Let $\mathbf{r}:[a,b]\to \mathcal{C}$ be a parametrization directly traversing $\mathcal{C}$. Then

$$\underbrace{\int_{\mathcal{C}} f(x,y,z)\,ds}_{\text{scalar line integral}} = \underbrace{\int_a^b f(\mathbf{r}(t))\,\|\mathbf{r}'(t)\|\,dt}_{\text{ordinary 1D-integral}}$$

Rmk) ds is often called arc-length differential. This theorem relates ds to the parametric differential dt.

Ex Compute $\int_{\mathcal{C}} (x+y+z)\,ds$ over the helix $\mathcal{C}$ parametrized by $\mathbf{r}(t) = \langle \cos(t), \sin(t), t\rangle$, $0 \le t \le 4\pi$.

Sol) Since

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1\rangle \implies \|\mathbf{r}'(t)\| = \sqrt{2},$$

we get

$$\int_{\mathcal{C}} (x+y+z)\,ds = \int_0^{4\pi} (\cos t + \sin t + t)\sqrt{2}\,dt$$

$$= 8\sqrt{2}\pi^2.$$

Ex Integrate $f(x,y) = x$ along the graph $\mathcal{C}$ of $y = x^2$, $0 \leq x \leq 1$.

Sol) Since no parametrization is given, we will begin by choosing one. An obvious choice is:

$$\mathbb{r}(t) = \langle t, t^2 \rangle, \qquad 0 \leq t \leq 1.$$

Then

$$\mathbb{r}'(t) = \langle 1, 2t \rangle \Rightarrow \| \mathbb{r}'(t) \| = \sqrt{1+4t^2},$$

and so,

$$\int_{\mathcal{C}} x \, ds = \int_0^1 t\sqrt{1+4t^2}\, dt$$

$$= \left[\frac{1}{12}(1+4t^2)^{3/2} \right]_0^1$$

$$= \frac{1}{12}\left(5^{3/2} - 1 \right).$$

Rmk) Any parametrizations satisfying the condition of THM can be used