Note 11

Section 17.1 Vector Fields

1 Intro.

- A vector field is an assignment of a vector to each point. Mathematically, it is a vector-valued function on a domain.

- A vector field in $\mathbb{R}^3$ is often denoted by

$$\mathbb{F}(x,y,z) = \langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)\rangle,$$

or alternatively,

$$= F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k},$$

where $\mathbf{i} = \langle 1,0,0\rangle$, $\mathbf{j} = \langle 0,1,0\rangle$, $\mathbf{k} = \langle 0,0,1\rangle$.

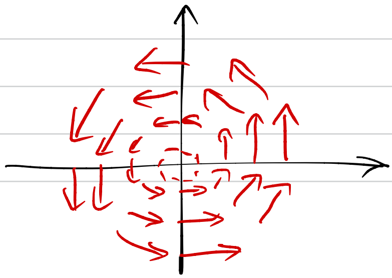
- Similarly, a vector field in $\mathbb{R}^2$ is often denoted by

$$\mathbb{F}(x,y) = \langle F_1(x,y), F_2(x,y)\rangle$$

$$= F_1\,\mathbf{i} + F_2\,\mathbf{j},$$

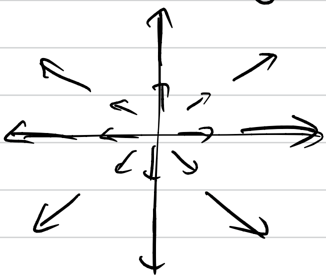
where $\mathbf{i} = \langle 1,0\rangle$, $\mathbf{j} = \langle 0,1\rangle$.

<u>Ex</u> $\mathbb{F}(x,y) = \langle -y, x\rangle$ looks like

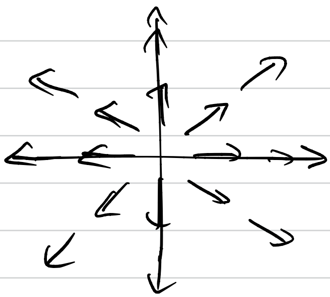


- Special kind of vector fields: we say that a vector field $\mathbb{F}$ is
  - **unif** vector field if $\|\mathbb{F}(P)\| = 1$ for any P.
  - **radial** vector field if
    $\mathbb{F}(P)$ is parallel to $\overrightarrow{OP}$, and
    $\|\mathbb{F}(P)\|$ depends only on $\|\overrightarrow{OP}\|$.

Ex • $\mathbb{F}(x,y) = \langle 3x, 3y \rangle$ is radial.



• $e_r(x,y) = \langle \frac{x}{r}, \frac{y}{r} \rangle = \langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \rangle$
is unit radial.



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2 Operations on Vector Field

- We define two important operations on VFs.
- Recall: del operator

$$\nabla = \begin{cases} \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle & \text{in 2D} \\ \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle & \text{in 3D.} \end{cases}$$

① Gradient of a scalar function f:

▸ In 2D, $\nabla f = \left\langle \frac{\partial}{\partial x} f, \frac{\partial}{\partial y} f \right\rangle$

▸ In 3D, $\nabla f = \left\langle \frac{\partial}{\partial x} f, \frac{\partial}{\partial y} f, \frac{\partial}{\partial z} f \right\rangle$.

② Divergence of a VF $\mathbb{F}$ in 3D:

$$\operatorname{div}(\mathbb{F}) = \nabla \cdot \mathbb{F} = \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3.$$

③ Curl of a VF $\mathbb{F}$ in 3D:

$$\operatorname{curl}(\mathbb{F}) = \nabla \times \mathbb{F} = \begin{vmatrix} \mathbb{i} & \mathbb{j} & \mathbb{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}$$

$$= \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle.$$

- Will study the meanings of div & curl later.
- <u>Note</u> Both div & curl satisfy linearity:

  If $\mathbb{F}, \mathbb{G}$: VF in $\mathbb{R}^3$, c : constant, then

$$\operatorname{div}(\mathbb{F}+\mathbb{G}) = \operatorname{div}(\mathbb{F}) + \operatorname{div}(\mathbb{G})$$
$$\operatorname{div}(c\mathbb{F}) = c\operatorname{div}(\mathbb{F})$$

and

$$\operatorname{curl}(\mathbb{F}+\mathbb{G}) = \operatorname{curl}(\mathbb{F}) + \operatorname{curl}(\mathbb{G})$$
$$\operatorname{curl}(c\mathbb{F}) = c\cdot\operatorname{curl}(\mathbb{F}).$$

$\boxed{\text{Ex}}$ $\operatorname{div}(e^x y,\ z-x,\ xz^2)$

$$= \frac{\partial}{\partial x}(e^x y) + \frac{\partial}{\partial y}(z-x) + \frac{\partial}{\partial z}(xz^2)$$

$$= e^x y + 0 + 2xz$$

$$= e^x y + 2xz.$$

$\boxed{\text{Ex}}$ $\operatorname{curl}(F_1(x,y),\ F_2(x,y),\ 0)$

$$= \left\langle \cancel{\frac{\partial}{\partial y}0} - \cancel{\frac{\partial}{\partial z}F_2}, \cancel{\frac{\partial}{\partial z}F_1} - \cancel{\frac{\partial}{\partial x}0}, \frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1 \right\rangle$$

$$= \left\langle 0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle.$$

$\boxed{\text{Caution}}$ Although we borrowed dot / cross product notations to conveniently write div / curl, they are NOT the actual dot / cross product. In particular,

$$\nabla \cdot \mathbb{F} \neq \underbrace{\mathbb{F} \cdot \nabla}, \qquad \nabla \times \mathbb{F} \neq \underbrace{\mathbb{F} \times \nabla}.$$

what this even mean?

3 Conservative VFs

- Gradient makes a scalar fn f into the vector field ∇f.

<u>Q</u> Can we do this backward?

- A vector field $\mathbb{F}$ is called **conservative** if we can find a scalar function f s.t.

$$\mathbb{F} = \nabla f.$$

In this context, f is called the **potential function** of $\mathbb{F}$.

<u>Q</u> Are all VFs conservative? <u>A</u> No.

$\boxed{\text{THM}}$ (Curl of a conservative VF)

- In 2D,

$$\begin{pmatrix} \mathbb{F} = \langle F_1, F_2 \rangle \\ \text{conservative} \end{pmatrix} \implies \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

- In 3D,

$$\begin{pmatrix} \mathbb{F} = \langle F_1, F_2, F_3 \rangle \\ \text{conservative} \end{pmatrix} \implies \text{curl}(\mathbb{F}) = 0.$$

$$\iff \begin{cases} \dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y} \\ \dfrac{\partial F_3}{\partial y} = \dfrac{\partial F_2}{\partial z} \\ \dfrac{\partial F_1}{\partial z} = \dfrac{\partial F_3}{\partial x} \end{cases}$$

<u>Rmk)</u> 2D case can be regarded as a special case of 3D, by considering $\langle F_1(x,y), F_2(x,y), 0 \rangle$.

$\boxed{\text{Ex}}$ If $f = e^x + 2yz^2$ and $\mathbb{F} = \nabla f$, then

$$\Rightarrow \mathbb{F} = \langle e^x, 2z^2, 4yz \rangle$$

$$\Rightarrow \text{curl}(\mathbb{F}) = \langle 4z - 4z, 0 - 0, 0 - 0 \rangle$$

$$= \mathbb{0}.$$

□

In general,

$$\operatorname{curl}(\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left\langle \underbrace{\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}}_{=0}, \underbrace{\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}}_{=0}, \cdots \right\rangle \quad \square$$

Rmk What we have just prove may be put into a diagram:

$$\begin{tikzcd} 1D \arrow[r, "grad"] \arrow[rr, bend right, "\text{curl of grad} = 0"'] & 3D \arrow[r, "curl"] & 3D \end{tikzcd}$$

$\boxed{\text{Ex}}$ $\mathbb{F} = \langle xy, \frac{x^2}{2}, zy \rangle$ is not conservative, because

$$\frac{\partial F_2}{\partial z} = \frac{\partial}{\partial z}\left(\frac{x^2}{2}\right) = 0 \neq z = \frac{\partial}{\partial y}(zy) = \frac{\partial F_3}{\partial y}.$$

$\boxed{\text{Ex}}$ Let $\mathbb{F} = \langle 3y^2+z, 6xy, x \rangle$. This is a conservative VF. Then find a potential of $\mathbb{F}$.

Sol) Let f be a potential of $\mathbb{F}$. Then

$$\frac{\partial f}{\partial x} = 3y^2 + z$$

$$\Rightarrow f = \int (3y^2+z)\,dx + \boxed{\text{something indep. of } x}$$