Note 10

16.5. Applications of Multiple Integrals

Recall that multiple integrals are defined as the limit of Riemann sums, and so, they can be understood as an ideal version of "sum of small quantities". And, although they are often understood as content (area/volume/hypervolume) of a region in a higher dimension, this is only one possible interpretation of integral. In general, the meaning of integral is inherited from that of the small quantities being summed up.

For instance, if $\mathcal{D}$ is a region in the plane and it is partitioned into subregions $\mathcal{D}_1, \cdots, \mathcal{D}_N$, each of which is so small that f is almost constant over it. Then by chooseing arbitrary point P_j from each subregion $\mathcal{D}_j$,

$$\iint_{\mathcal{D}} f(x)\,\mathrm{d}x \approx \sum_{j=1}^{N} f(P_j)\,\Delta A_j, \qquad \Delta A_j = \operatorname{area}(\mathcal{D}_j).$$

Now, under the geometric interpretation, this quantity is understood as the (signed) volume of the region under the graph $z = f(x, y)$ over $\mathcal{D}$, and this is because we regard $f(x)$ as height function so that each summand

$$\underbrace{\boxed{f(P_j)\,\Delta A_j}}_{\text{volume}} = \underbrace{\boxed{f(P_j)}}_{\text{height}} \times \underbrace{\boxed{\Delta A_j}}_{\text{base area}}$$

represents the volume of the narrow box with the base area ΔA_j and height $f(P_j)$. This way, the integral inherits the meaning as "volume" from that of each small volumes $f(P_j)\,\Delta A_j$.

1. Total amount

In this part, we instead consider the interpretation

$$[\text{quantity}] = [\text{density per area}] \times [\text{area}] \qquad \text{or} \qquad [\text{density per volume}] \times [\text{volume}].$$

Indeed, writing δ in place of f so as to emphasize its role as a density function,

$$[\text{quantity over } \mathcal{D}_j] = \underbrace{\boxed{\delta(P_j)}}_{\text{density}} \times \underbrace{\boxed{\Delta A_j}}_{\text{area}}$$

and hence these sum up to the approximate total quantity

$$[\text{total quantity}] = \sum_{j=1}^{N} [\text{quantity over } \mathcal{D}_j] \approx \sum_{j=1}^{N} \delta(P_j)\,\Delta A_j \approx \iint_{\mathcal{D}} \delta(x, y)\,\mathrm{d}A,$$

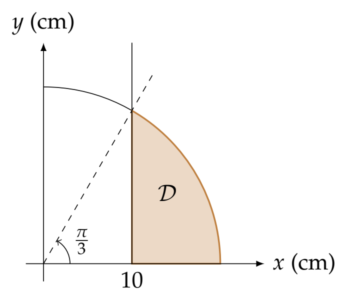
which will then become the true equality by passing to the limit. Accordingly, we have

The **total amount** of a quantity distributed over a region according to a density function is the integral of the density function over the region.

- In 2D, [total amount] $= \iint_{\mathcal{D}} \delta(x,y)\,\mathrm{d}A$.
- In 3D, [total amount] $= \iint_{\mathcal{W}} \delta(x,y,z)\,\mathrm{d}V$.

Example (Total Mass)

Compute the total mass of the plate $\mathcal{D}$ in the following figure, assuming a mass density of $f(x,y) = x^2/(x^2+y^2)$ g/cm^2.



Solution. Since both $f(x,y)$ and $\mathcal{D}$ are easily described in polar coordinates, we compute the total mass using the Change of Variables Formula in polar coordinates.

- The outer circle has radius $10\sec(\frac{\pi}{3}) = 20$ cm.
- The line $x = 10$ is converted to $r = 10\sec\theta$ in polar coordinates.
- These determine $\mathcal{D}$ as the radially simple region $0 \le \theta \le \frac{\pi}{3}$, $10\sec\theta \le r \le 20$.
- The function is written as $f(x,y) = f(r\cos\theta, r\sin\theta) = \cos^2\theta$ in polar coordinates.

So it follows that

$$\begin{aligned}
[\text{total mass}] &= \iint_{\mathcal{D}} f(x,y)\,\mathrm{d}A \\
&= \int_0^{\frac{\pi}{3}} \int_{10\sec\theta}^{20} r\cos^2\theta\,\mathrm{d}r\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{3}} (200\cos^2\theta - 50)\,\mathrm{d}\theta \\
&= 25\sqrt{3} + \frac{50\pi}{3} \quad \text{in gram.}
\end{aligned}$$

□

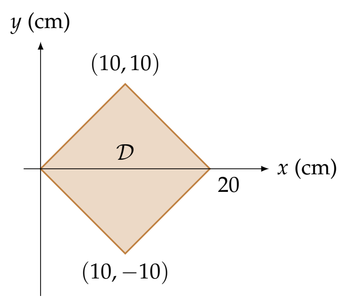
2. Average density/quantity

The **average value** $\overline{f}$ of a function f over the region is:

- In 2D, $\overline{f} = \dfrac{[\text{total amount of } f]}{[\text{total area}]} = \dfrac{\iint_{\mathcal{D}} f(x,y)\,\mathrm{d}A}{\iint_{\mathcal{D}} 1\,\mathrm{d}A}$.
- In 3D, $\overline{f} = \dfrac{[\text{total amount of } f]}{[\text{total volume}]} = \dfrac{\iint_{\mathcal{W}} f(x,y,z)\,\mathrm{d}V}{\iint_{\mathcal{W}} 1\,\mathrm{d}V}$.

Example (Average Temperature)

Suppose that $f(x,y) = 300 + axy$ in kelvin describes the temperature map of the metal plate in the following figure, where $a = 10$ K/cm^2. Compute the average temperature across $\mathcal{D}$.



Solution. The area of $\mathcal{D}$ is 200 cm^2, and so,

$$[\text{avg. temp.}] = \frac{1}{200}\iint_{\mathcal{D}} f(x,y)\,\mathrm{d}A.$$

The integral can be computed in several ways.

- Splitting the region along the line $x = 10$, the integral becomes

$$\begin{aligned}\iint_{\mathcal{D}} f(x,y)\,\mathrm{d}A &= \int_0^{10}\int_{-x}^{x}(300+10xy)\,\mathrm{d}y\mathrm{d}x + \int_{10}^{20}\int_{x-20}^{20-x}(300+10xy)\,\mathrm{d}y\mathrm{d}x \\ &= \int_0^{10} 2x(300+10xy)\,\mathrm{d}x + \int_{10}^{20}(40-2x)(300+10xy)\,\mathrm{d}x \\ &= 60000\ \mathrm{K}\cdot\mathrm{cm}^2.\end{aligned}$$

- Alternatively, we may invoke the linear map $T(u,v) = (u+v, v-u)$. Since $T(1,0) = (1,-1)$ and $T(0,1) = (1,1)$, T maps the square $\mathcal{R} = [0,10]\times[0,10]$ to the region $\mathcal{D}$. Also, $\mathrm{Jac}(T) = 2$.

So by the Change of Variables Formula,

$$\begin{aligned}\iint_{\mathcal{D}} f(x,y)\,\mathrm{d}A &= \iint_{\mathcal{R}} f(u+v, v-u)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\mathrm{d}v \\ &= \int_0^{10}\int_0^{10} (600 + 20v^2 - 20u^2)\,\mathrm{d}u\mathrm{d}v \\ &= 60000\ \mathrm{K}{\cdot}\mathrm{cm}^2.\end{aligned}$$

- Yet another way is to observe that $\iint_{\mathcal{D}} xy\,\mathrm{d}A = 0$ by the symmetry around the x-axis, and so,

$$\iint_{\mathcal{D}} f(x,y)\,\mathrm{d}A = \iint_{\mathcal{D}} 300\,\mathrm{d}A = 300 \cdot \mathrm{area}(\mathcal{D}).$$

Both computations show that the average temperature is 300 K. □

3. Centroid of a region

The **centroid** of a region is the average position of all points of the region.

- In 2D, the centroid of the region $\mathcal{D}$ is

$$\langle \overline{x}, \overline{y} \rangle = \frac{\iint_{\mathcal{D}} \langle x, y\rangle\,\mathrm{d}A}{\iint_{\mathcal{D}} 1\,\mathrm{d}A}$$

- In 3D, the centroid of the region $\mathcal{W}$ is

$$\langle \overline{x}, \overline{y}, \overline{z} \rangle = \frac{\iiint_{\mathcal{W}} \langle x, y, z\rangle\,\mathrm{d}V}{\iiint_{\mathcal{W}} 1\,\mathrm{d}V}$$

Example (Centroid of a quarter circle)

Find the centroid of the quarter circle $\mathcal{D} = \{(x,y) : x^2 + y^2 \le R^2,\ x \ge 0,\ y \ge 0\}$.

Solution. Recall that $\mathrm{area}(\mathcal{D}) = \frac{\pi R^2}{4}$. Then its centroid $(\overline{x}, \overline{y})$ is

$$\begin{aligned}\overline{x} = \frac{4}{\pi R^2}\iint_{\mathcal{D}} x\,\mathrm{d}A &= \frac{4}{\pi R^2}\int_0^{\frac{\pi}{2}}\int_0^R r^2\cos\theta\,\mathrm{d}r\mathrm{d}\theta \\ &= \frac{4R}{3\pi}\int_0^{\frac{\pi}{2}} \cos\theta\,\mathrm{d}\theta = \frac{4R}{3\pi}.\end{aligned}$$

Similar computation shows that $\overline{y} = 4R/3\pi$ as well, which intuitively makes sense because the region is symmetric through the line $y = x$. □

Example (Centroid is compatible with linear map)

Suppose that:

- T is a linear map with $\mathrm{Jac}(T) \neq 0$.
- $\mathcal{D}_0$ is a region in the uv-plane with the centroid $(\overline{u}, \overline{v})$.

Then show that the centroid of $T(\mathcal{D}_0)$ is $T(\overline{u}, \overline{v})$.

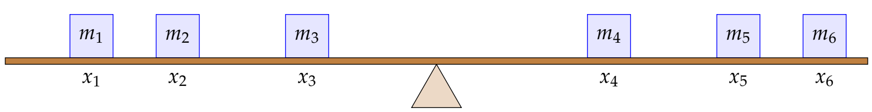
Solution. Write $T = (Au + Cv, Bu + Dv)$ for constants A, B, C, D and recall that $\mathrm{Jac}(T) = AD - BC$ is constant and $\mathrm{area}(T(\mathcal{D}_0)) = |\mathrm{Jac}(T)|\,\mathrm{area}(\mathcal{D}_0)$ holds. Now write $(\overline{x}, \overline{y})$ for the centroid of $T(\mathcal{D}_0)$. Then by the Change of Variables Formula,

$$
\begin{aligned}
\overline{x} &= \frac{1}{\mathrm{area}(T(\mathcal{D}_0))} \iint_{T(\mathcal{D}_0)} x \,\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{\cancel{|\mathrm{Jac}(G)|}\,\mathrm{area}(\mathcal{D}_0)} \iint_{T(\mathcal{D}_0)} (Au + Cv) \cancel{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|} \,\mathrm{d}u\mathrm{d}v \\
&= A\overline{u} + C\overline{v},
\end{aligned}
$$

which is exactly the x-coordinate of $T(\overline{u}, \overline{v})$. A similar computation shows that $\overline{y} = B\overline{u} + D\overline{v}$ as well, and therefore the desired claim follows. □

4. Center of Mass

Suppose we have a plate with masses $m_1, \cdots, m_N$ at positions $x_1, \cdots, x_N$:



At which point should we place the fulcrum in order to balance the plate? The law of physics tells that the fulcrum should be placed at the weighted average x_{CM} of x_i's:

$$
x_{\mathrm{CM}} = \frac{x_1 m_1 + \cdots + x_N m_N}{m_1 + \cdots + m_N}.
$$

Now, instead of point masses, consider the mass which is continuously distributed according the density function $\delta(x)$. In such case, the idea of Riemann sum kicks in and the above weighted average becomes

$$
x_{\mathrm{CM}} = \frac{\int x\delta(x)\,\mathrm{d}x}{\int \delta(x)\,\mathrm{d}x}.
$$

This idea generalizes to higher dimensions:

The **center of mass** (COM) of a region is the weighted average of positions.

- In 2D, the COM of the region $\mathcal{D}$ is

$$\langle x_{\text{CM}}, y_{\text{CM}} \rangle = \frac{\iint_{\mathcal{D}} \langle x, y \rangle \delta(x,y) \, \mathrm{d}A}{\iint_{\mathcal{D}} \delta(x,y) \, \mathrm{d}A}$$

- In 3D, the COM of the region $\mathcal{W}$ is

$$\langle x_{\text{CM}}, y_{\text{CM}}, z_{\text{CM}} \rangle = \frac{\iiint_{\mathcal{W}} \langle x, y, z \rangle \delta(x,y,z) \, \mathrm{d}V}{\iiint_{\mathcal{W}} \delta(x,y,z) \, \mathrm{d}V}.$$

If the density function is constant (i.e., the density is uniform), then the COM coincides with the centroid.