Note 9

16.6. Change of Variables

Last time, we shifted focus to another perspective, where one coordinate system is related to another by a map that actually transform points and sets. Then we analyzed how the points and sets transform under linear maps. In particular, we learned that areas change under linear maps in a uniform manner. We generalize this observation to nonlinear maps and establish the Change of Variables Formula.

Jacobian determinant and Change of Variables Formula

The **Jacobian determinant** (or simply the Jacobian) of a map $G(u,v) = (x(u,v), y(u,v))$ is the determinant

$$\mathrm{Jac}(G) = \frac{\partial(x,y)}{\partial(u,v)} := \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

In general, $\mathrm{Jac}(G)$ is a function of u, v. So, borrowing the function notation, it makes sense to write $\mathrm{Jac}(G)(P) = \mathrm{Jac}(G)(u,v)$ to denote the Jacobian of G evaluated at the point $P = (u,v)$.

Example (Jacobian of a linear map)

For the linear map $G(u,v) = (Au + Cv, Bu + Dv)$, its Jacobian is constant with the value

$$\mathrm{Jac}(G) = \begin{vmatrix} A & C \\ B & D \end{vmatrix} = AD - BC.$$

Consequently, for any rectangle $\mathcal{R}$ in the uv-plane,

$$\mathrm{area}(G(\mathcal{R})) = |\mathrm{Jac}(G)|\,\mathrm{area}(\mathcal{R}).$$

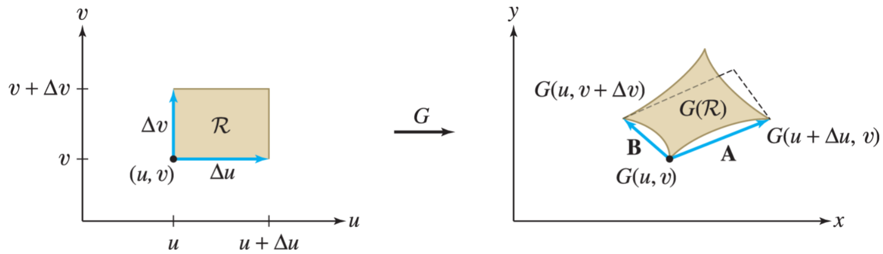
The above example does not generalize directly to nonlinear maps. However, if we restrict to a very small region $\mathcal{D}_0$ in the uv-plane, then G behaves almost linearly, and so, an approximate result

$$\mathrm{area}(G(\mathcal{D}_0)) \approx |\mathrm{Jac}(G)(P)|\,\mathrm{area}(\mathcal{D}_0)$$

holds, where $P \in \mathcal{D}_0$. A bit more precise formulation takes the following form: If $P \in \mathcal{D}_0$ and G is a "nice" map, then

$$|\mathrm{Jac}(G)(P)| = \lim_{|\mathcal{D}_0| \to 0} \frac{\mathrm{area}(G(\mathcal{D}_0))}{\mathrm{area}(\mathcal{D}_0)}.$$

Here, $|\mathcal{D}_0| \to 0$ indicates the limit as the diameter of $\mathcal{D}$ tends to zero.



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Now suppose in addition that f is a continuous function over $G(\mathcal{D}_0)$. If $\mathcal{D}_0$ is still very small, then $G(\mathcal{D}_0)$ would also be very small, rendering f almost constant over the region $G(\mathcal{D}_0)$. Therefore we anticipate:

$$\begin{aligned}\iint_{G(\mathcal{D}_0)} f(x,y)\,\mathrm{d}x\mathrm{d}y &\approx f(P)\,\mathrm{area}(G(\mathcal{D}_0)) \\ &\approx f(P)\,|\mathrm{Jac}(G)(P)|\,\mathrm{area}(\mathcal{D}_0) \\ &\approx \iint_{\mathcal{D}_0} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\mathrm{d}v\end{aligned} \tag{16.1}$$

Although the above heuristics are only approximately true for small domains, this generalizes to a precise identity under reasonable assumptions.

Theorem (Change of Variables Formula)

Let $G : \mathcal{D}_0 \to \mathbb{R}^2$ satisfy:

- G is one-to-one at least on the interior of $\mathcal{D}_0$.
- G is C^1, i.e., G has continuous partial derivatives.

If $f(x,y)$ is continuous, then

$$\iint_{G(\mathcal{D}_0)} f(x,y)\,\mathrm{d}x\mathrm{d}y = \iint_{\mathcal{D}_0} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\mathrm{d}v$$

Here, $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ denotes the absolute value of the Jacobian.

Remark. The assumption that G is C^1 is more or less a convenience that makes the statement and proof of theorem easier. That being said, for the purpose of this course, we will rarely bother with this technicality.

However, the assumption that G is one-to-one is crucial. In analogy with summation, this condition is essential for preventing "over-counting". For example, consider the polar coordinate map $G(r,\theta) = (r\cos\theta, r\sin\theta)$.

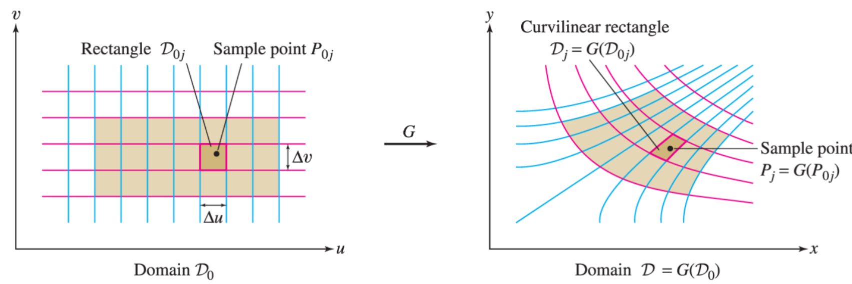
- G maps the rectangle $\mathcal{R} = [0,1] \times [0,2\pi]$ to the unit disk, and G is indeed one-to-one on the interior of $\mathcal{R}$. Note that G is not one-to-one on all of $\mathcal{R}$, since the entire left side $\{0\} \times [0,2\pi]$ of $\mathcal{R}$ is mapped to a single point $(0,0)$ in the xy-plane, and the top and bottom side of $\mathcal{R}$ is "glued" in the image since $G(r,0) = G(r,2\pi)$. Such over-counting on the boundary of $\mathcal{R}$ does not harm the validity of the Change of Variables Formula.

- On the other hand, G also maps the rectangle $\mathcal{R} = [0,1] \times [0,6\pi]$ to the unit disk. In this case, however, this is done in such a way that each point of the disk is "coverd" at least three times. Since the Change of Variables Formula does not capture this over-counting, the formula is inapplicable in this case.

Sketch of Proof. We may decompose $\mathcal{D}_0$ into small subdomains $\mathcal{D}_{0,1}, \cdots, \mathcal{D}_{0,N}$. Then applying the approximation (16.1) to each subdomain $\mathcal{D}_{0,j}$, we expect:

$$\iint_{G(\mathcal{D}_{0,j})} f(x,y)\,\mathrm{d}x\mathrm{d}y \approx \iint_{\mathcal{D}_{0,j}} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\mathrm{d}v \tag{16.2}$$

But since G is one-to-one, $G(\mathcal{D}_{0,1}), \cdots, G(\mathcal{D}_{0,N})$ comprise a non-overlapping decomposition of the region $G(\mathcal{D})$. (If G is not one-to-one, then some of these subregions may overlap. This is where the one-to-one assumption is used.)



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So we have

$$\iint_{G(\mathcal{D}_0)} f(x,y)\,\mathrm{d}x\mathrm{d}y = \sum_{j=1}^{N} \iint_{G(\mathcal{D}_{0,j})} f(x,y)\,\mathrm{d}x\mathrm{d}y.$$

Approximating each summand using (16.2),

$$\begin{aligned}\iint_{G(\mathcal{D}_0)} f(x,y)\,\mathrm{d}x\mathrm{d}y &\approx \sum_{j=1}^{N} \iint_{\mathcal{D}_{0,j}} f(x(u,v),y(u,v))\,|\mathrm{Jac}(G)(u,v)|\,\mathrm{d}u\mathrm{d}v \\ &= \iint_{\mathcal{D}_0} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\mathrm{d}v.\end{aligned}$$

This approximation can be shown to become a precise identity when passed to the limit as the diameters of subdivided regions tend to zero. □

It is sometimes useful to represent the Jacobian of $(x,y) = G(u,v)$ in terms of x,y. This can be done by the following formula.

Theorem

If $(u,v) = G^{-1}(x,y)$ denotes the inverse map of $(x,y) = G(u,v)$, then

$$[\text{Jac}(G) \text{ evaluated at } (u,v)] = \frac{1}{[\text{Jac}(G^{-1}) \text{ evaluated at } (x,y)]}$$

provided $\text{Jac}(G^{-1}) \neq 0$. This may be written in the suggestive form:

$$\frac{\partial(x,y)}{\partial(u,v)} = \left[\frac{\partial(u,v)}{\partial(x,y)}\right]^{-1}.$$

Finally, it is worth to mention that the Change of Variables Formula extends to 3D *mutatis mutandis*, if we define the Jacobian of $(x,y,z) = G(u,v,w)$ by

$$\text{Jac}(G) = \frac{\partial(x,y,z)}{\partial(u,v,w)} := \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Examples

Example (Polar coordinates revisited)

Let $G(r,\theta) = (r\cos\theta, r\sin\theta)$ be the polar coordinate map. Then

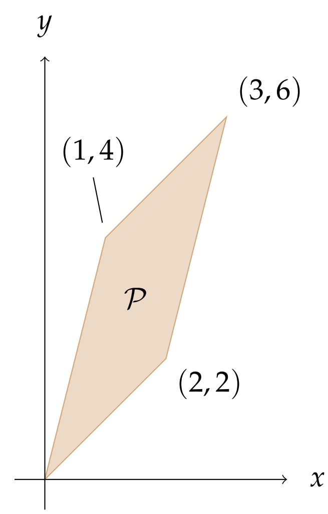
$$\text{Jac}(G) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = (\cos\theta)(r\cos\theta) - (-r\sin\theta)(\sin\theta) = r.$$

So the Change of Variables Formula in polar coordinate is recovered.

Example (Exercise 24)

Find a linear map T that maps $[0,1] \times [0,1]$ to the parallelogram $\mathcal{P}$ in the xy-plane with vertices $(0,0)$, $(2,2)$, $(1,4)$, $(3,6)$. Then calculate the double integral of e^{2x-y} over $\mathcal{P}$ via change of variables.

Solution.



Step 1. Define the map. $\mathcal{P}$ is spanned by two vectors $\langle 2, 2\rangle$ and $\langle 1, 4\rangle$. So we may choose T as a linear map that maps $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$ to those vectors. One such choice is

$$T(1,0) = (2,2) \qquad \text{and} \qquad T(0,1) = (1,4),$$

which then determines T as

$$T(u,v) = (2u + v, 2u + 4v).$$

Step 2. Compute the Jacobian. Since T is linear, its Jacobian is easily computed as

$$\mathrm{Jac}(G) = \begin{vmatrix} 2 & 1 \\ 2 & 4 \end{vmatrix} = 6.$$

Step 3. Express $f(x, y)$ in terms of the new variables. Plugging $(x, y) = T(u, v)$ to the function $f(x, y) = e^{2x-y}$, we get

$$f(x,y) = f(T(u,v)) = f(2u+v, 2u+4v) = e^{2u-2v}.$$

Step 3. Apply the Change of Variables Formula. Applying the change of variables shows that

$$\mathrm{d}x\mathrm{d}y = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\mathrm{d}v = 6\mathrm{d}u\mathrm{d}v,$$

and so,

$$\begin{aligned}\iint_{\mathcal{R}} e^{2x-y}\,\mathrm{d}x\mathrm{d}y &= \iint_{[0,1]\times[0,1]} e^{2u-2v} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\mathrm{d}v \\ &= \int_0^1 \int_0^1 6e^{2u-2v}\,\mathrm{d}u\mathrm{d}v = \frac{3}{2}(e^2 - 1)(1 - e^{-2}).\end{aligned}$$

□

Example (Exercise 35)

Calculate $\iint_{\mathcal{D}} e^{9x^2+4y^2}\,\mathrm{d}x\mathrm{d}y$, where $\mathcal{D}$ is the interior of the ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 \le 1$.

Solution.

It is natural to choose $(x, y) = G(u, v)$ so that G parametrizes $\mathcal{D}$ using a square in the uv-plane. Motivated by the polar coordinates, we pick

$$G(u,v) = (2u\cos v, 3u\sin v).$$

If we set $\mathcal{R} = [0,1] \times [0, 2\pi]$, then G is one-to-one on the interior of $\mathcal{R}$ and $G(\mathcal{R}) = \mathcal{D}$. Moreover,

$$\operatorname{Jac}(G) = \begin{vmatrix} 2\cos v & -2u\sin v \\ 3\sin v & 3u\cos v \end{vmatrix} = 6u \qquad \text{and} \qquad e^{9x^2+4y^2} = e^{36u^2}.$$

Combining altogether, we get

$$\iint_{\mathcal{D}} e^{9x^2+4y^2}\,\mathrm{d}x\mathrm{d}y = \iint_{\mathcal{R}} e^{36u^2} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\mathrm{d}v = \int_0^{2\pi}\int_0^1 6ue^{36u^2}\,\mathrm{d}u\mathrm{d}v = \frac{\pi}{6}(e^{36}-1).$$

$\square$

Example (Exercise 41)

Compute $I = \iint_{\mathcal{D}} (x^2 - y^2)\,\mathrm{d}x\mathrm{d}y$, where

$$\mathcal{D} = \{(x,y) : 2 \le xy \le 4,\ 0 \le x - y \le 3,\ x \ge 0,\ y \ge 0\}.$$

1st Solution. It is natural to choose the map $(x,y) = G(u,v)$ such that $u = xy$ and $v = x - y$. Then $\mathcal{D}$ is the image of the rectangle $\mathcal{R} = [2,4] \times [0,3]$ under G. Also, by solving these equations,

$$x = \frac{v + \sqrt{v^2+4u}}{2}, \qquad y = \frac{-v + \sqrt{v^2+4u}}{2}.$$

Here, the signs are chosen so that this indeed defined a map from the rectangle $\mathcal{R}$ to $\mathcal{D}$ which is one-to-one on the interior of $\mathcal{R}$. Then the Jacobian of G is computed as

$$\operatorname{Jac}(G) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{\sqrt{v^2+4u}} & \dfrac{v+\sqrt{v^2+4u}}{2\sqrt{v^2+4u}} \\ \dfrac{1}{\sqrt{v^2+4u}} & \dfrac{v-\sqrt{v^2+4u}}{2\sqrt{v^2+4u}} \end{vmatrix} = -\frac{1}{\sqrt{v^2+4u}}.$$

Also, plugging $(x,y) = G(u,v)$ to the integrand $x^2 - y^2$ gives

$$x^2 - y^2 = v\sqrt{v^2+4u}.$$

So by the Change of Variables Formula,

$$I = \iint_{\mathcal{R}} v\sqrt{v^2+4u}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\mathrm{d}v = \int_0^3\int_2^4 v\,\mathrm{d}u\mathrm{d}v = 9.$$

$\square$

2nd Solution. Again we choose the map $(x,y) = G(u,v)$ by the relations $u = xy$ and $v = x - y$, so that $\mathcal{D}$ is the image of the rectangle $\mathcal{R} = [2,4] \times [0,3]$ under G. In this case, however, we directly work with x, y coordinates before converting everything in terms of u, v. Then the Jacobian of G^{-1} is computed as

$$\operatorname{Jac}(G^{-1}) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 1 & -1 \end{vmatrix} = -(x+y).$$

So by the Change of Variables Formula together with the relation $\operatorname{Jac}(G) = [\operatorname{Jac}(G^{-1})]^{-1}$,

$$\begin{aligned} I &= \iint_{\mathcal{R}} (x^2 - y^2) \left| \frac{\partial(u,v)}{\partial(x,u)} \right|^{-1} \mathrm{d}u\mathrm{d}v = \iint_{\mathcal{R}} (x^2 - y^2) \cdot \frac{1}{x+y} \,\mathrm{d}u\mathrm{d}v \\ &= \iint_{\mathcal{R}} \underbrace{(x-y)}_{=v} \,\mathrm{d}u\mathrm{d}v = \int_0^3 \int_2^4 v \,\mathrm{d}u\mathrm{d}v = 9. \end{aligned}$$

$\square$