Note 7

16.4. Integration in Polar, Cylindrical, and Spherical Coordinates

Last time, we learned how to convert a double integral in rectangular coordinates to another one in polar coordinates.

Theorem: Change of variables formula in polar coordinates

Let $\mathcal{D}$ a radially simple region

$$\mathcal{D} : \quad \alpha_1 \le \theta \le \alpha_2 \quad \text{and} \quad r_1(\theta) \le r \le r_2(\theta).$$

Then

$$\iint_{\mathcal{R}} f(x,y) \boxed{\mathrm{d}A} = \int_{\alpha_1}^{\alpha_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) \boxed{r\mathrm{d}r\mathrm{d}\theta}.$$

This formula was obtained by noting that, partitioning the region $\mathcal{D}$ in polar coordinates, then the area ΔA_{ij} of each small polar subrectangle $\mathcal{R}_{ij}$ is approximately $r_j \Delta r_j \Delta\theta_i$.

Today, we will apply similar ideas to the triple integral in two coordinate systems, namely the *cylindrical coordinates* and *spherical coordinates*.

(1) Cylindrical coordinates This is a straightforward generalization of polar coordinates to 3D, by adding z coordinate to polar coordinate. Indeed, a point in the space can be written by

$$\begin{pmatrix} (x,y,z) \\ \text{in rectangular coordinates} \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} (r,\theta,z) \\ \text{in cylindrical coordinates,} \end{pmatrix}$$

where the conversion formula is given by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

By the Fubini's theorem, we immediately obtain

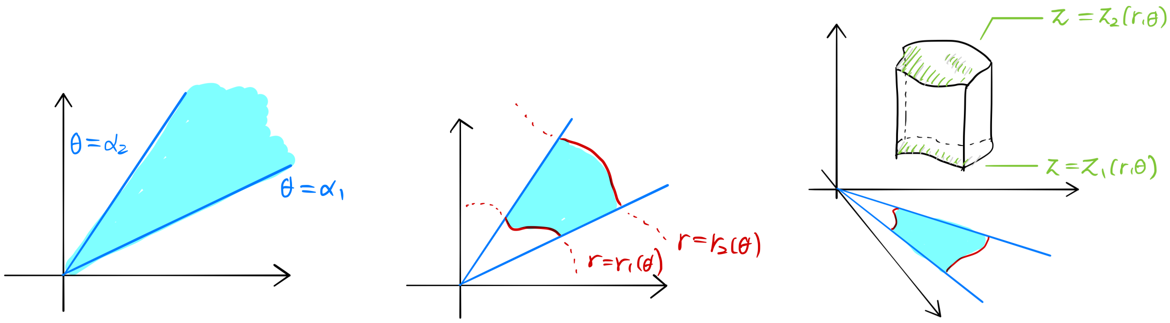
Theorem: Change of variables formula in cylindrical coordinates

Let $\mathcal{W}$ be described, in cylindrical coordinates, by

$$\mathcal{W} : \quad \alpha_1 \le \theta \le \alpha_2, \quad r_1(\theta) \le r \le r_2(\theta), \quad z_1(r,\theta) \le z \le z_2(r,\theta).$$

Then

$$\iiint_{\mathcal{W}} f(x,y,z) \boxed{\mathrm{d}V} = \int_{\alpha_1}^{\alpha_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r\cos\theta, r\sin\theta, z) \boxed{r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta}.$$



The planar region given by $\alpha_1 \leq \theta \leq \alpha_2$.

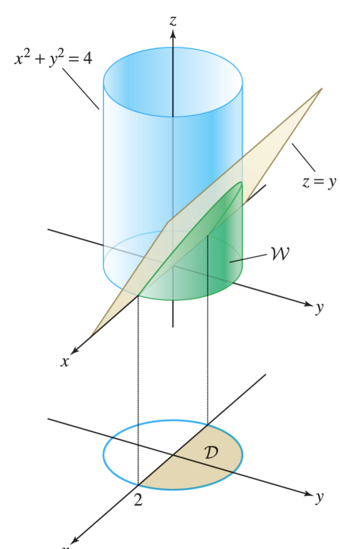
The planar region $\mathcal{D}$ given by $\alpha_1 \leq \theta \leq \alpha_2$, $r_1(\theta) \leq r \leq r_2(\theta)$.

The solid region $\mathcal{W}$ given by $\alpha_1 \leq \theta \leq \alpha_2$, $r_1(\theta) \leq r \leq r_2(\theta)$, $z_1(r,\theta) \leq z \leq z_2(r,\theta)$.

Figure 1: Progressively carving out the solid region $\mathcal{W}$ described in the theorem.

In a typical question, $\mathcal{W}$ will not be given explicitly, either because it is described in another coordinates system or because it is only implicitly defined. That being said, we will first need to express $\mathcal{W}$ in cylindrical coordinates, and then use the change of variables formula to evaluate the triple integral.

Exercise. *Compute the integral of $f(x,y,z) = z$ over the region $\mathcal{W}$ within the cylinder $x^2 + y^2 \leq 4$, where $0 \leq z \leq y$.*



Rogawski et al., *Multivariable Calculus*, 4e, © 2019 W. H. Freeman and Company

Solution. We first express $\mathcal{W}$ in cylindrical coordinates.

- The condition $x^2 + y^2 \leq 4$ converts to $0 \leq r \leq 2$.
- From the condition, we obtain $y \geq 0$. This converts to $0 \leq \theta \leq \pi$.

The above conditions altogether tells that $\mathcal{W}$ projects onto the semidisk given by $0 \leq \theta \leq \pi$ and $0 \leq r \leq 2$. Then

- The condition $0 \leq z \leq y$ converts to $0 \leq z \leq r \sin\theta$.

Therefore

$$\mathcal{W} : \quad 0 \leq \theta \leq \pi, \quad 0 \leq r \leq 2, \quad 0 \leq z \leq r\sin\theta.$$

Also, $f(x,y,z) = f(r\cos\theta, r\sin\theta, z) = z$. Then by the change of variables formula,

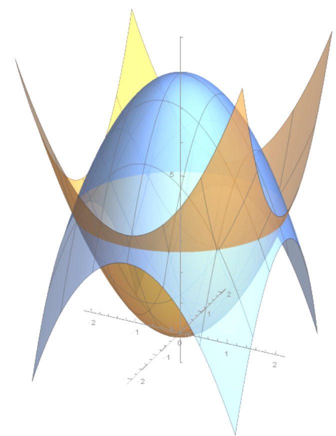
$$\begin{aligned}\iiint_{\mathcal{W}} f(x,y,z)\,\mathrm{d}V &= \int_0^{\pi}\int_0^2\int_0^{r\sin\theta} zr\,\mathrm{d}z\mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{\pi}\int_0^2 \frac{(r\sin\theta)^2}{2}\cdot r\,\mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{\pi} 2\sin^2\theta\,\mathrm{d}\theta = \pi.\end{aligned}$$

The last step follows from the formula $\int \sin^2\theta\,\mathrm{d}\theta = \frac{1}{2}(\theta - \sin\theta\cos\theta) + C.$ □

Exercise. *Use cylindrical coordinates to calculate* $\iiint_{\mathcal{W}} z\sqrt{x^2+y^2}\,\mathrm{d}V$, *where*

$$\mathcal{W}\ :\quad x^2+y^2 \le z \le 8-(x^2+y^2).$$

Solution. We first express $\mathcal{W}$ in cylindrical coordinates.



- The inequality converts to $r^2 \le z \le 8-r^2$. This naturally determines the bounds of z.
- There is no restriction to θ. Or, geometrically speaking, the region is axially symmetric. So we get $0 \le \theta \le 2\pi$.
- For $r^2 \le 8-r^2$ to hold, we must have $r \le 2$.

Summarizing, we get

$$\mathcal{W}\ :\quad 0 \le \theta \le 2\pi, \quad 0 \le r \le 2, \quad r^2 \le z \le 8-z^2.$$

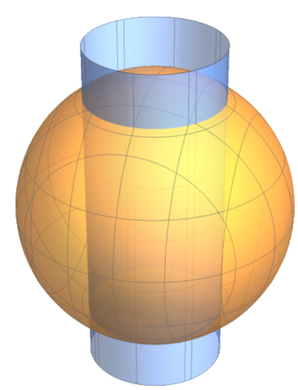
Also, the integrand becomes $z\sqrt{x^2+y^2} = zr$. So by the theorem,

$$\begin{aligned}\iiint_{\mathcal{W}} z\sqrt{x^2+y^2}\,\mathrm{d}V &= \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} zr\cdot r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{2\pi}\int_0^2 r^2\left(\frac{(8-r^2)^2}{2}-\frac{(r^2)^2}{2}\right)\mathrm{d}r\mathrm{d}\theta \\ &= \left(\int_0^{2\pi}\mathrm{d}\theta\right)\left(\int_0^2(32r^2-8r^4)\,\mathrm{d}r\right) = \frac{1024\pi}{15}.\end{aligned}$$

□

Exercise. *Use cylindrical coordinates to find the volume of a sphere of radius* $2a$ *from which a central cylinder of radius* a *has been removed.*

Solution. Call the region $\mathcal{W}$. We observe:



- $\mathcal{W}$ projects onto the annulus $a \le r \le 2a$.
- Since $\mathcal{W}$ lies inside of the sphere, we have $r^2+z^2 \le (2a)^2$. This determines the bounds of z as $-\sqrt{4a^2-r^2} \le z \le \sqrt{4a^2-r^2}$.

Combining altogether,

$$\mathcal{W}\ :\quad 0 \le \theta \le 2\pi, \quad a \le r \le 2a, \quad -\sqrt{4a^2-r^2} \le z \le \sqrt{4a^2-r^2}.$$

From this, we get

$$\begin{aligned}\mathrm{volume}(\mathcal{W}) = \iiint_{\mathcal{W}}\mathrm{d}V &= \int_0^{2\pi}\int_a^{2a}\int_{-\sqrt{4a^2-r^2}}^{\sqrt{4a^2-r^2}} r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{2\pi}\int_a^{2a} 2r\sqrt{4a^2-r^2}\,\mathrm{d}r\mathrm{d}\theta.\end{aligned}$$

The inner integral can be easily computed by substituting $u = 4a^2 - r^2$. Then $\mathrm{d}u = -2r\mathrm{d}r$ and hence

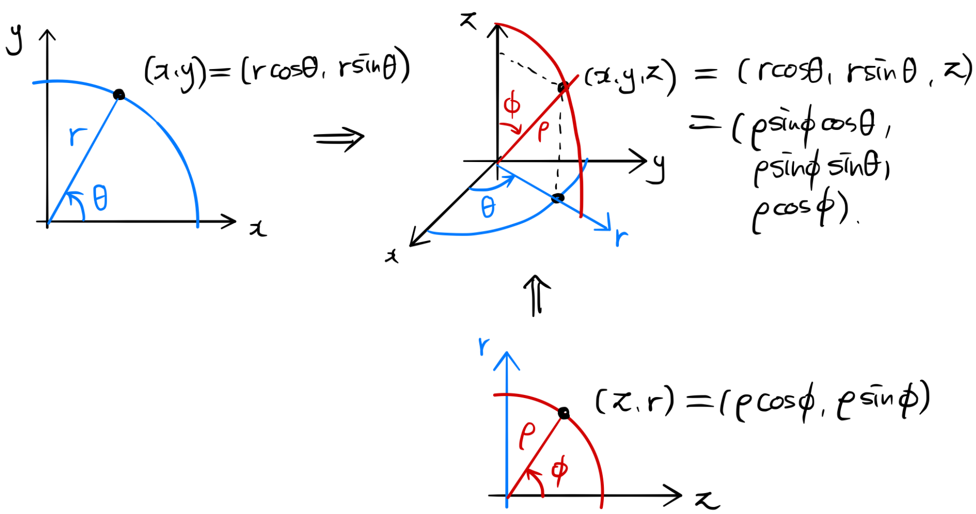
$$\int_{r=a}^{r=2a} 2r\sqrt{4a^2 - r^2}\,\mathrm{d}r = \int_{u=0}^{u=3a^2} \sqrt{u}\,\mathrm{d}u = \left[\frac{2}{3}u^{\frac{3}{2}}\right]_{u=0}^{u=3a^2} = 2\sqrt{3}a^3.$$

Therefore

$$\mathrm{volume}(\mathcal{W}) = \int_0^{2\pi} 2\sqrt{3}a^3\,\mathrm{d}\theta = 4\pi\sqrt{3}a^3.$$

□

(2) Spherical coordinates. Roughly speaking, it is obtained by applying the idea of polar coordinates to zr-plane in cylindrical coordinates. That is, we want to write $z = \rho\cos\phi$ and $r = \rho\sin\phi$.



This gives

$$\begin{pmatrix} (x,y,z) \\ \text{in rectangular coordinates} \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} (\rho,\theta,\phi) \\ \text{in spherical coordinates,} \end{pmatrix}$$

where the conversion formula is given by

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi.$$

The "grid surfaces" are as follows:

- $\rho = \rho_0$ is the sphere of radius ρ_0 centered at the origin.
- $\phi = \phi_0$ is a cone around the z-axis.
- $\theta = \theta_0$ is a half-plane starting from the z-axis.

Now we want to derive the change of variables formula for spherical coordinates. To this end, note that the small "spherical wedge", corresponding to

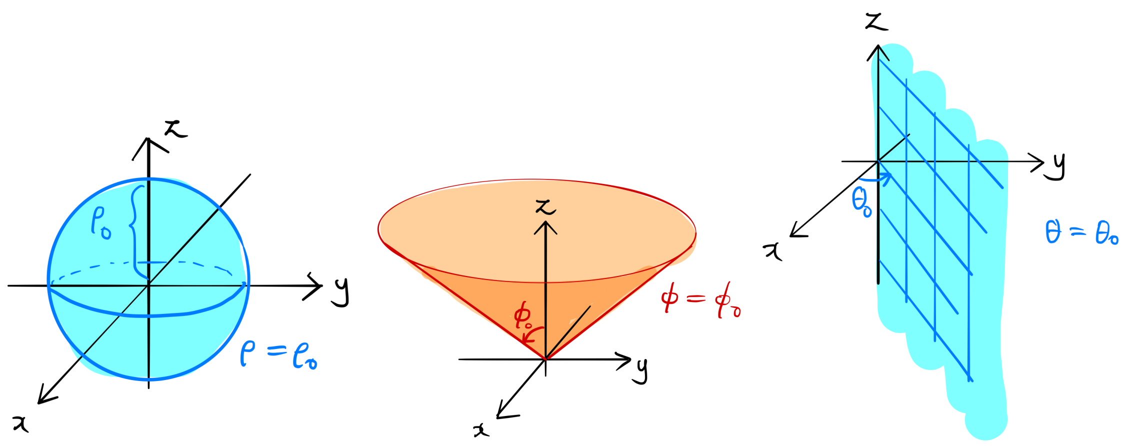
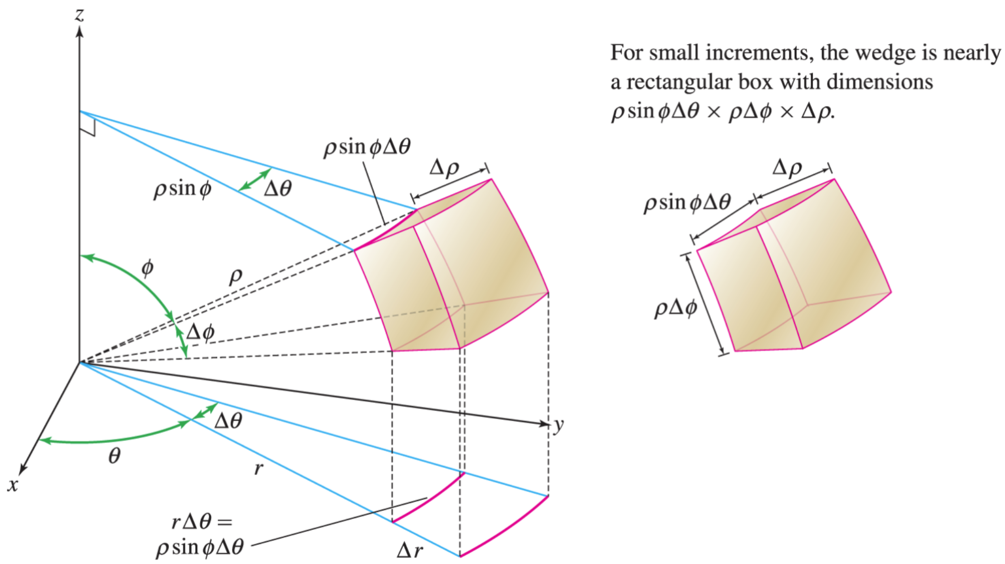


Figure 2: Grid surfaces

the region $[\rho, \rho + \Delta\rho] \times [\theta, \theta + \Delta\theta] \times [\phi, \phi + \Delta\phi]$ in spherical coordinates



is approximately a box with the volume

$$(\Delta\rho)(\rho\Delta\phi)(\rho \sin\phi\Delta\theta) = \rho^2 \sin\phi\Delta\rho\Delta\phi\Delta\theta.$$

So, similarly as before, we get

Theorem: Change of variables formula in spherical coordinates

Let $\mathcal{W}$ be a *centrally simple region*, i.e., described in spherical coordinates by

$$\mathcal{W} \; : \quad \theta_1 \leq \theta \leq \theta_2, \quad \phi_1 \leq \phi \leq \phi_2, \quad \rho_1(\theta, \phi) \leq \rho \leq \rho_2(\theta, \phi).$$

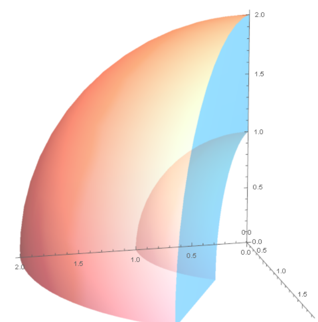
Then

$$\iiint_{\mathcal{W}} f(x,y,z)\,\boxed{\mathrm{d}V} = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1(\theta,\phi)}^{\rho_2(\theta,\phi)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\boxed{\rho^2\sin\phi\,\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta}.$$

Exercise. *Evaluate the integral of $f(x,y,z) = z$ over the region*

$$\mathcal{W}\ :\quad 0 \le \theta \le \frac{\pi}{3},\quad 0 \le \phi \le \frac{\pi}{2},\quad 1 \le \rho \le 2.$$

Solution. Plugging the conversion formula, we get $f(x,y,z) = \rho\cos\phi$. So by the change of variables formula,



$$\begin{aligned}\iiint_{\mathcal{W}} f(x,y,z)\,\mathrm{d}V &= \int_0^{\frac{\pi}{3}}\int_0^{\frac{\pi}{2}}\int_1^2 \rho^3\cos\phi\sin\phi\,\mathrm{d}z\mathrm{d}\phi\mathrm{d}\theta \\ &= \left(\int_0^{\frac{\pi}{3}}\mathrm{d}\theta\right)\left(\int_0^{\frac{\pi}{2}}\cos\phi\sin\phi\,\mathrm{d}\phi\right)\left(\int_1^2\rho^3\,\mathrm{d}\rho\right) \\ &= \frac{\pi}{3}\cdot\frac{1}{2}\cdot\frac{15}{4} = \frac{5\pi}{8}.\end{aligned}$$

□

Exercise. *Evaluate the integral of $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ over the region $\mathcal{W}$ given by $x^2+y^2+z^2 \le 2z$.*

Solution. We express $\mathcal{W}$ in spherical coordinates.

- The inequality can be rearranged to $x^2+y^2+(z-1)^2 \le 1$, which is the unit sphere centered at $(0,0,1)$. This entirely lies in the region $z \ge 0$, which is a centrally simple region $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi/2$.
- It remains to determine the range of ρ. Plugging the conversion formula to the inequality, we get $0 \le \rho \le 2\cos\phi$.

Summarizing,

$$\mathcal{W}\ :\quad 0 \le \theta \le 2\pi,\quad 0 \le \phi \le \pi/2,\quad 0 \le \rho \le 2\cos\phi.$$

Also, $f(x,y,z) = \rho$. So we get

$$\begin{aligned}\iiint_{\mathcal{W}} f(x,y,z)\,\mathrm{d}V &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2\cos\phi}\rho^3\sin\phi\,\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta \\ &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} 4\cos^4\phi\sin\phi\,\mathrm{d}\phi\mathrm{d}\theta = \int_0^{2\pi}\frac{4}{5}\,\mathrm{d}\theta = \frac{8\pi}{5}.\end{aligned}$$

In the third step, we utilized the formula $\int\cos^4\phi\sin\phi\,\mathrm{d}\phi = -\frac{1}{5}\cos^5\phi + C$, which can be easily derived by substituting $u = \cos\phi$. □