Note 3

16.2 Double integrals over more general regions

GOAL
- Extend the definition of double integral to the case of more general domains.
- Learn how to compute such integrals.

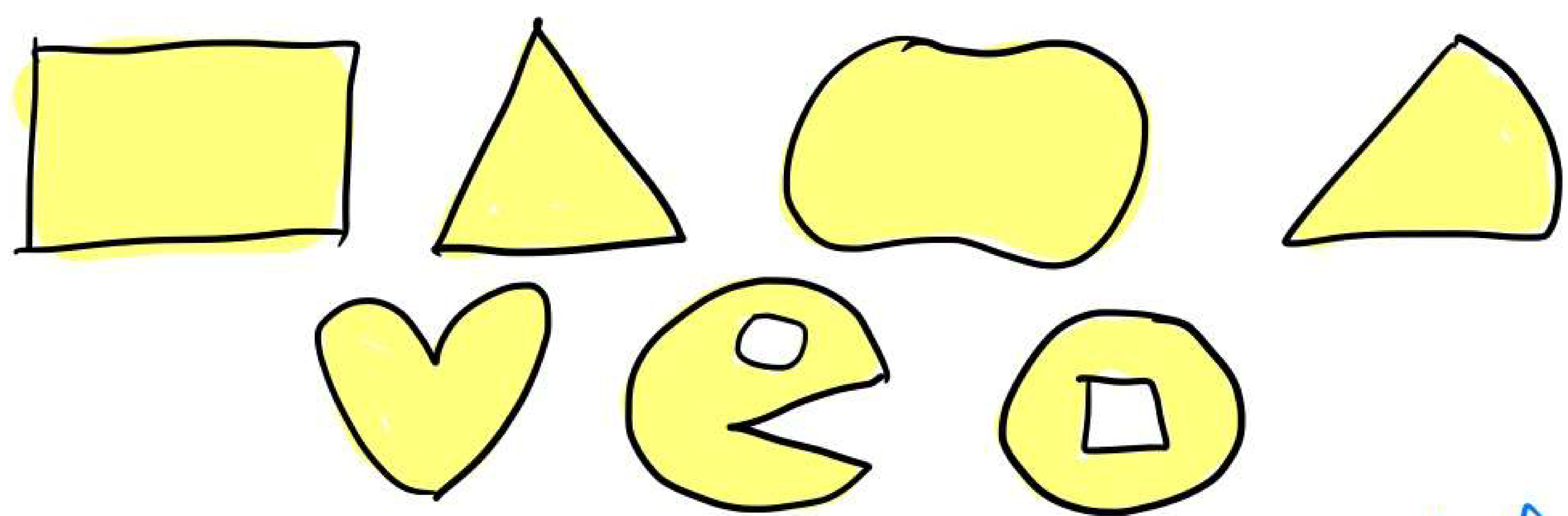
① Definition

$\boxed{\text{Assumption}}$ D is a "good" domain, in the sense that the boundary of D consists of finitely many piecewise-smooth simple closed curves.

(= finitely many smooth curved joined together) (= loops that do not self-intersect.)

(Don't worry about this technical detail!)

$\boxed{\text{Examples}}$



set of all real numbers.

$\boxed{\text{DEF}}$ If D : "good" domain, $f : D \to \mathbb{R}$, then for any rectangle R s.t. $D \subseteq R$, define

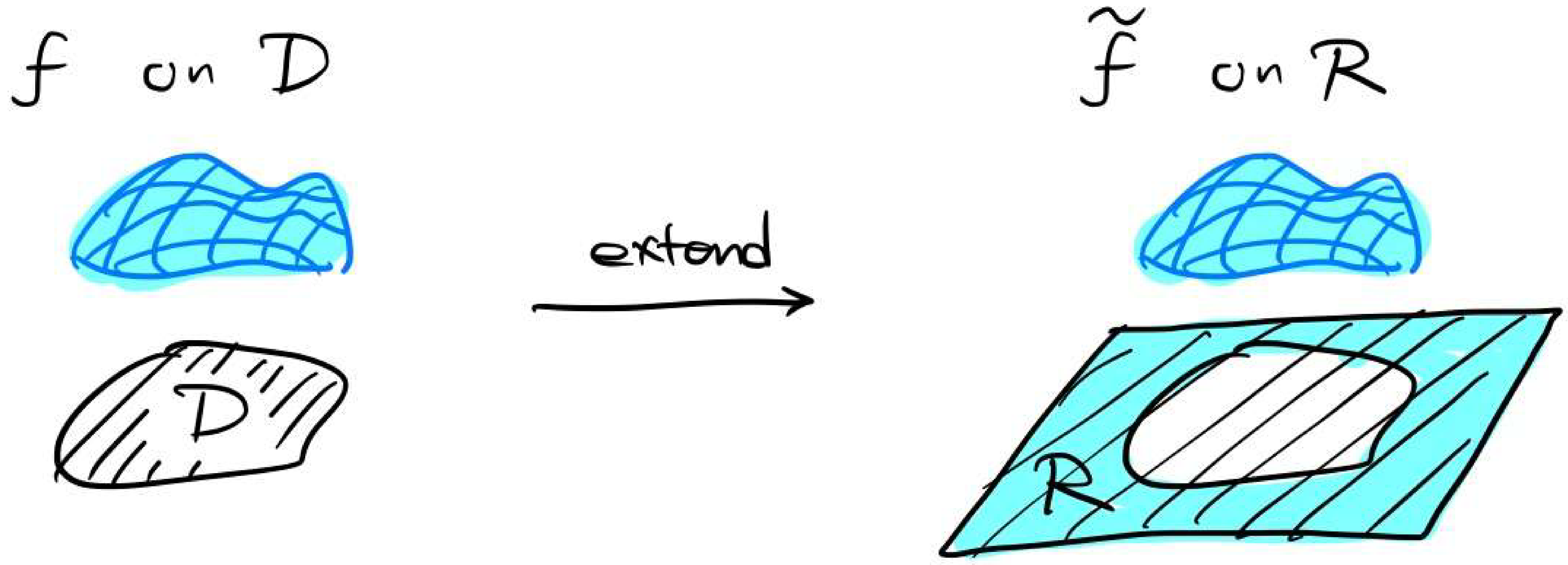
$$\tilde{f}(x,y) := \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{o/w} \end{cases}$$

and

$$\iint_D f(x,y)\, dA := \iint_R \tilde{f}(x,y)\, dA.$$

f is called integrable over D if the above integral exists.

Rmks)

- 

- The value of $\iint_D f(x,y)\,dA$ does not depend on the choice of R.

$\boxed{\text{THM 1}}$ If

(1) D : closed domain whose boundary is a simple closed piecewise-smooth curve (i.e., closed "good" domain without holes)

(2) $f : D \to \mathbb{R}$ continuous,

Then f is integrable over D.

$\boxed{\text{Facts}}$

(1) Double integral over D is linear.

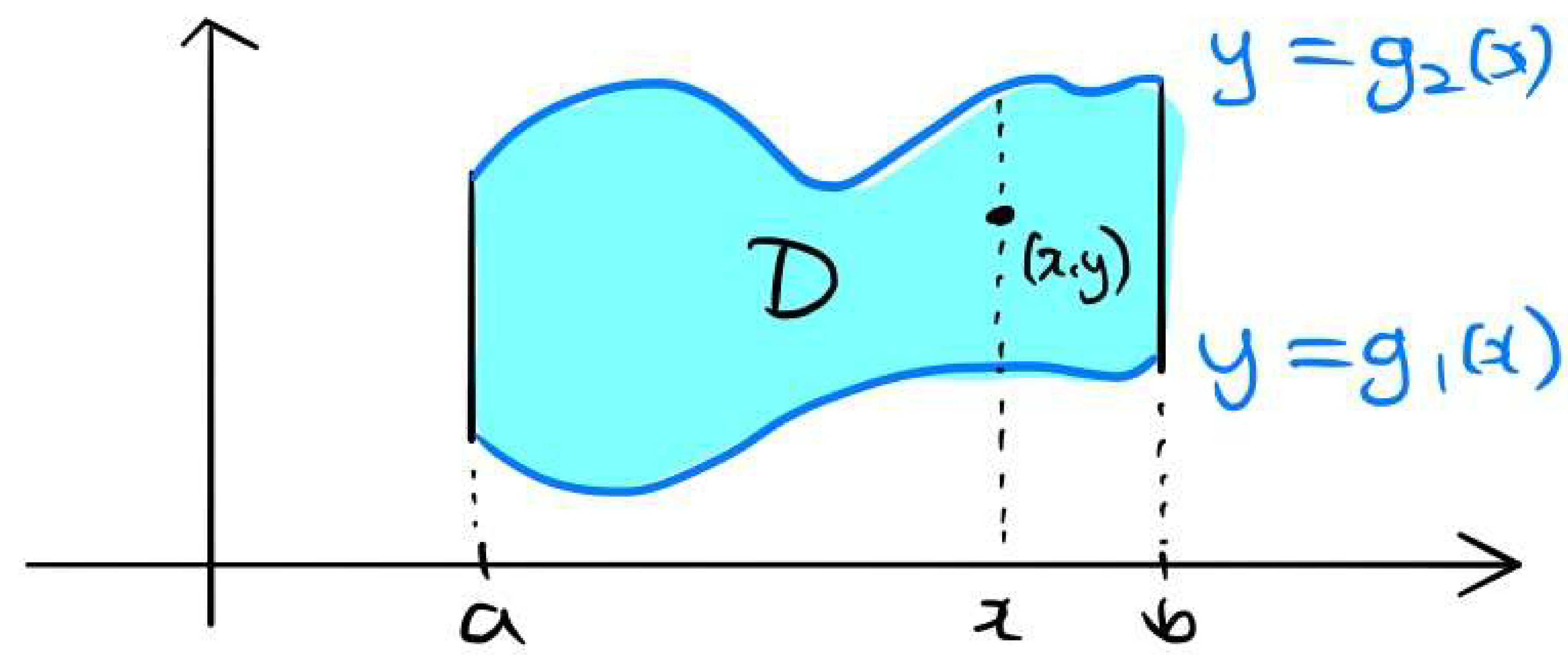
(2) $\iint_D 1 \, dA = [\text{area of } D]$.

② Simple domains

- Will consider special kinds of domains that are easier to deal with.

$\boxed{\text{DEF}}$ (1) D is vertically simple if it is of the form

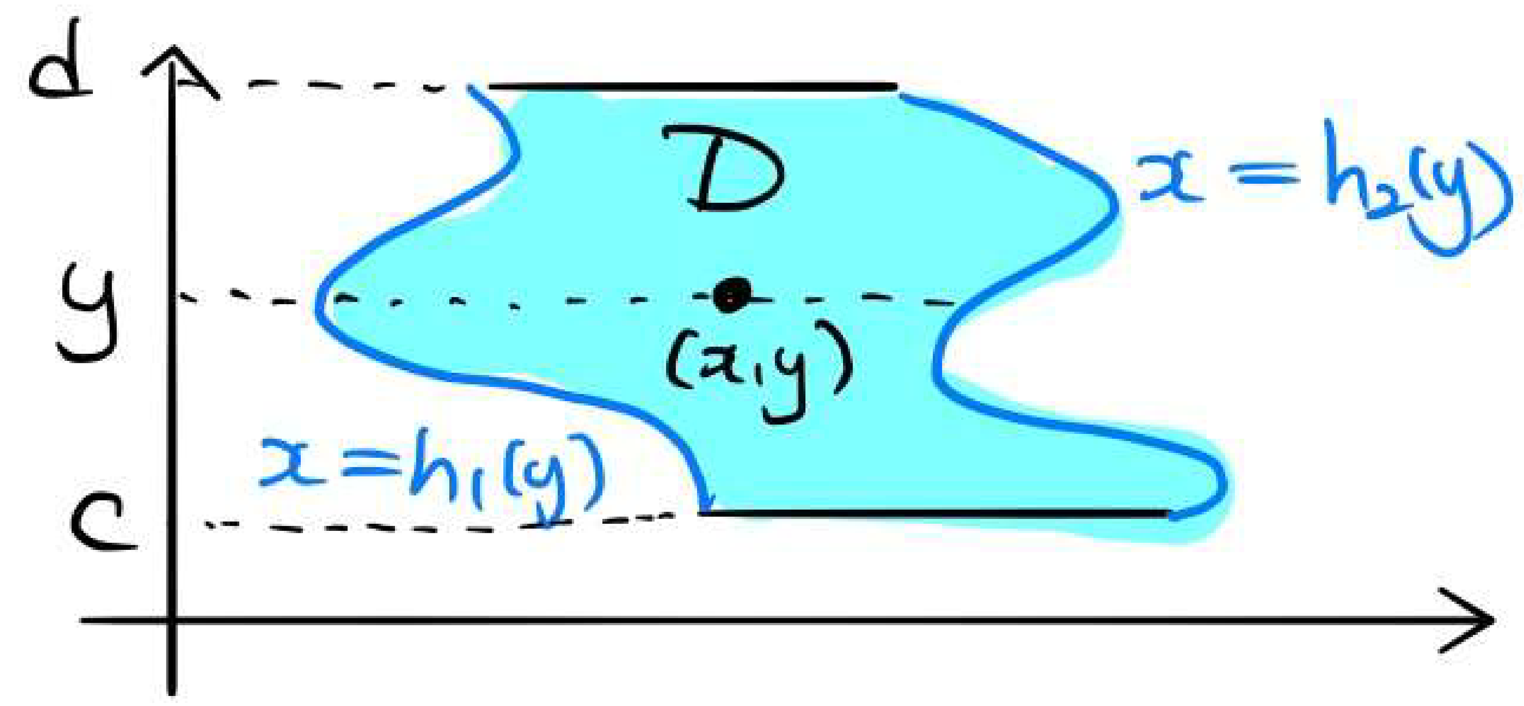
$$D = \{(x,y) : a \le x \le b,\ \ g_1(x) \le y \le g_2(x)\}$$



for some continuous ftns $g_1, g_2 : [a,b] \to \mathbb{R}$.

(2) D is horizontally simple if

$$D = \{(x,y) : c \le y \le d,\ \ h_1(y) \le x \le h_2(y)\}$$



for some continuous ftns $h_1, h_2 : [c,d] \to \mathbb{R}$.

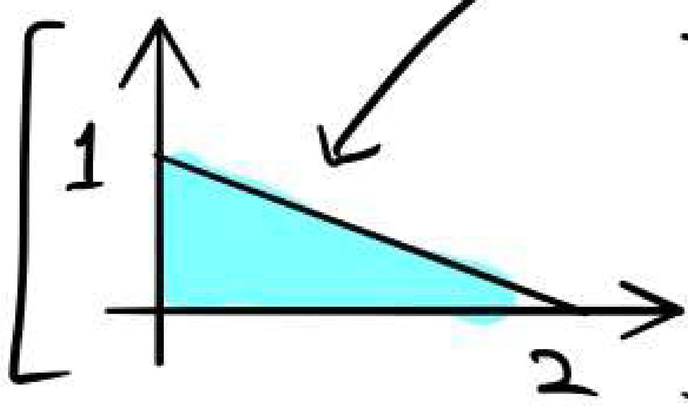
$\boxed{\text{THM 2}}$ (1) If D is vertically simple as in DEF. (1),

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

(2) If D is horizontally simple as in DEF. (2),

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

$y = 1 - \frac{x}{2}$, or $x = 2 - 2y$.

$\boxed{\text{Ex}}$ Let $D = \left[\right.$  $\left.\right]$. Compute $\iint_D x^2 y \, dA$.

1st Sol)

(Step 1) D is vertically simple :

$$D = \{ (x,y) : 0 \le x \le 2, \quad \overset{g_1}{0} \le y \le \overset{g_2}{1 - \tfrac{1}{2}x} \}$$

(Step 2) Apply THM 2 :

$$\iint_D x^2 y \, dA = \int_0^2 \int_0^{1-\frac{x}{2}} x^2 y \, dy dx.$$

(Step 3) Compute!

$$\int_0^{1-x/2} x^2 y \, dy = \left[x^2 \cdot \frac{y^2}{2} \right]_{y=0}^{1-\frac{x}{2}}$$

$$= x^2 \cdot \frac{1}{2}\left(1 - \frac{x}{2}\right)^2 - \cancel{x^2} \cdot 0$$

$$= \frac{x^2 (2-x)^2}{8}$$

$$\Rightarrow \iint_D x^2 y \, dA = \int_0^2 \frac{x^2(2-x)^2}{8} dx$$

$$= \frac{2}{15}.$$

$\square$

2nd Sol)

(Step 1) D is horizontally simple :

$$D = \{ (x,y) : 0 \le y \le 1, \quad \overset{h_1}{0} \le x \le \overset{h_2}{2-2y} \}$$

(Step 2) Apply THM 2 :

$$\iint_D f(x,y)\,dA = \int_0^1 \int_0^{2-2y} x^2 y \,dx\,dy$$

(Step 3) Compute!

$$\int_0^{2-2y} x^2 y \,dx = \left[\frac{x^3}{3} \cdot y \right]_{x=0}^{2-2y}$$

$$= \frac{(2-2y)^3}{3} \cdot y - \cancel{\frac{0^3}{3} \cdot y}$$

$$= \frac{8}{3}(1-y)^3 y .$$

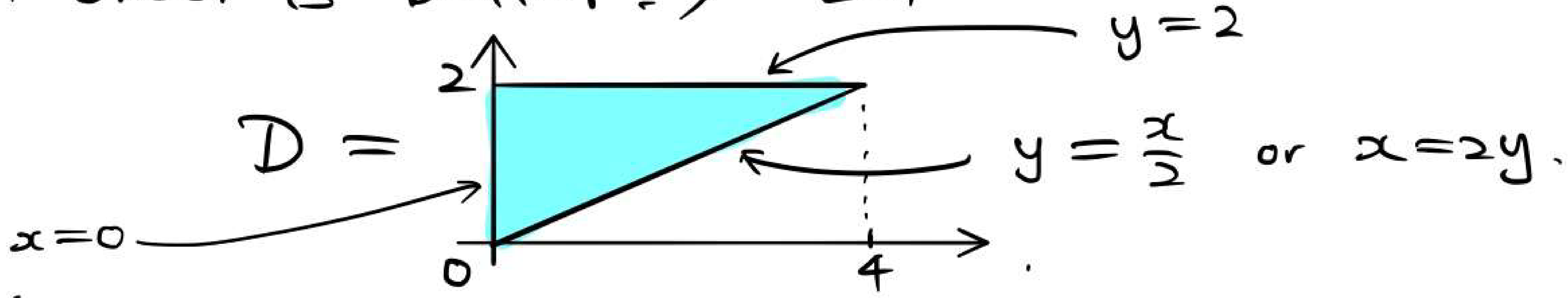
$$\Rightarrow \iint_D f(x,y)\,dA = \int_0^1 \frac{8}{3}(1-y)^3 y \,dy$$

$$= \frac{2}{15} .$$

□

$\boxed{\text{Ex}}$ (Which order is better?) Let

$D =$



Compute

$$I = \iint_D e^{y^2}\,dA .$$

<u>Sol)</u> Regarding D as vertically simple,

$$I = \int_0^4 \underbrace{\int_{\frac{x}{2}}^{2} e^{y^2}\,dy\,dx}_{} = \cdots ???$$

$\int e^{y^2} dy$ has no explicit expression!

Regarding D as horizontally simple,

$$I = \int_0^2 \int_0^{2y} e^{y^2}\,dx\,dy$$

$$= \int_0^2 \left[x \cdot e^{y^2} \right]_{x=0}^{2y} dy$$

$$= \int_0^2 2y\,e^{y^2}\,dy$$

$$= \left[e^{y^2} \right]_0^2 = e^4 - 1.$$

$\square$

$\boxed{\text{Ex}}$ (Changing the order of integration) Evaluate

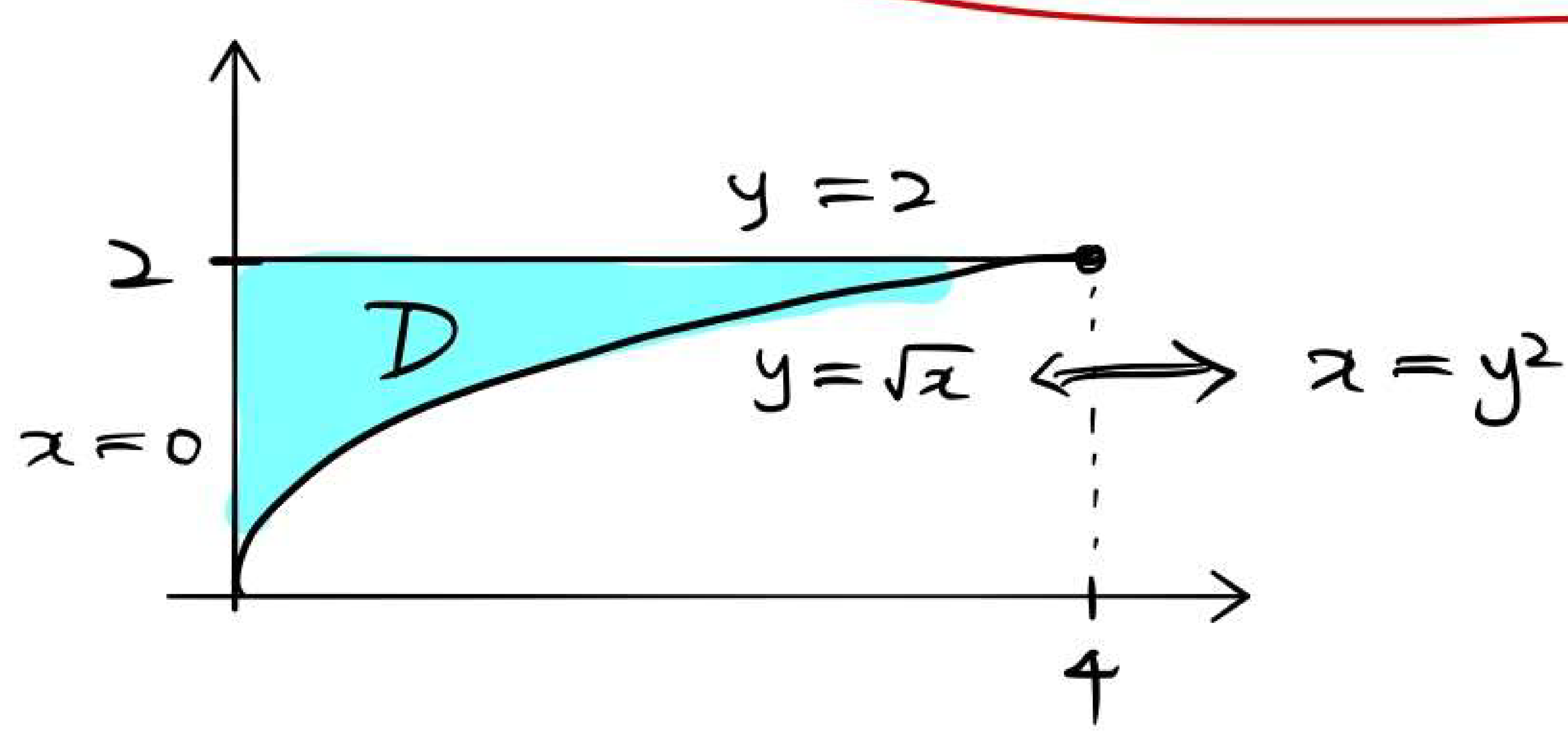
$$I = \int_0^4 \int_{\sqrt{x}}^2 \sin(y^3)\,dy\,dx.$$

by changing the order of integration.

<u>Rmk)</u> $\int \sin(y^3)\,dy$ has no explicit expression!

<u>Sol)</u> (Step 1) The domain of integration of I is specified by:

$$\begin{cases} 0 \le x \le 4 \\ \sqrt{x} \le y \le 2 \end{cases} \qquad \left(\int_0^4 \int_{\sqrt{x}}^2 \cdots\,dy\,dx \right)$$



As a horizontally simple region, D is specified by

$$\begin{cases} 0 \le y \le 2 \\ 0 \le x \le y^2. \end{cases}$$

So,

$$\begin{aligned} I &= \int_0^2 \int_0^{y^2} \sin(y^3)\, dx\, dy \\ &= \int_0^2 \left[x \cdot \sin(y^3) \right]_{x=0}^{y^2} dy \\ &= \int_0^2 y^2 \sin(y^3)\, dy \\ &= \left[-\frac{1}{3} \cos(y^3) \right]_0^2 = \frac{1}{3}(1-\cos(8)). \end{aligned}$$

$\square$

$\boxed{\text{THM3}}$ (Comparisons) Let f, g be integrable on D.

(a) If $f(x,y) \le g(x,y)$ for all $(x,y) \in D$,

$$\iint_D f(x,y)\, dA \le \iint_D g(x,y)\, dA$$

(b) If $m \le f(x,y) \le M$ for all $(x,y) \in D$,

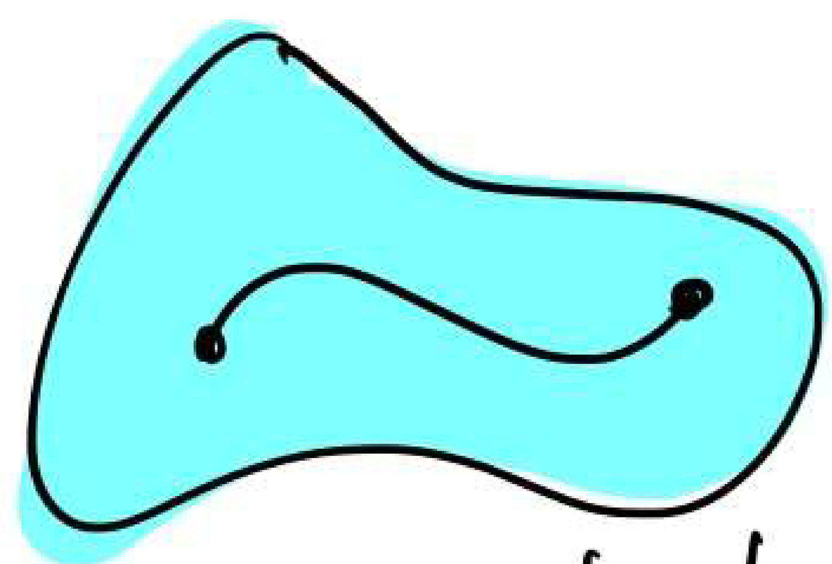
$$m \cdot \text{area}(D) \le \iint_D f(x,y)\, dA \le M \cdot \text{area}(D).$$

$\boxed{\text{THM4}}$ (Mean Value Theorem) Let D: closed, bounded, connected domain and $f: D \to \mathbb{R}$ continuous. Then

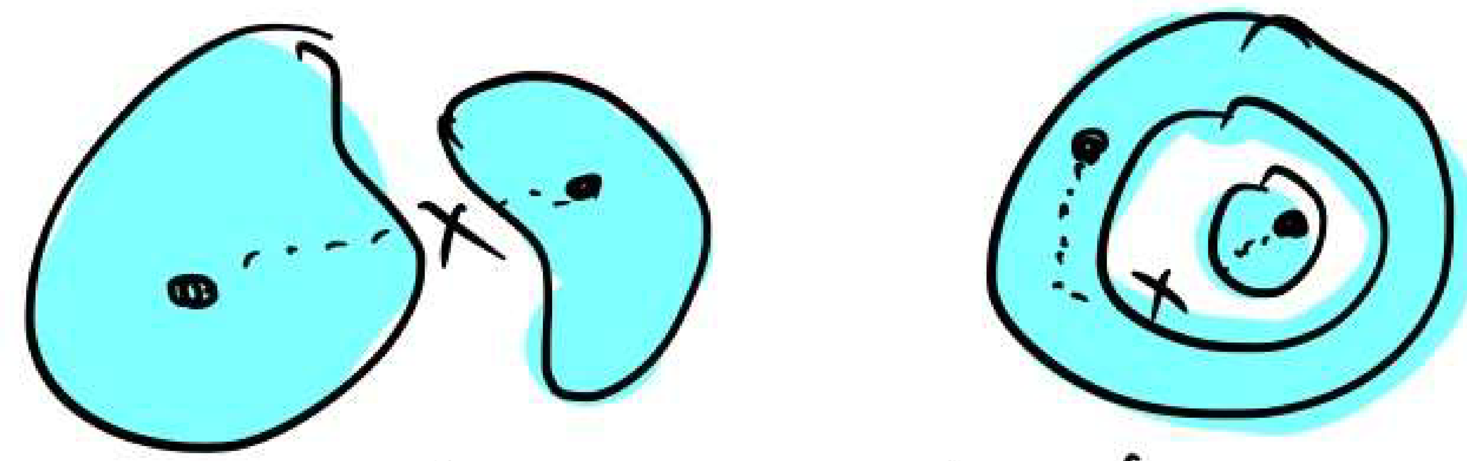
$$\iint_D f(x,y)\, dA = f(P) \cdot \text{area}(D)$$

for some $P \in D$.

Here, D is connected if any two points of D can be joined by a curve in D.



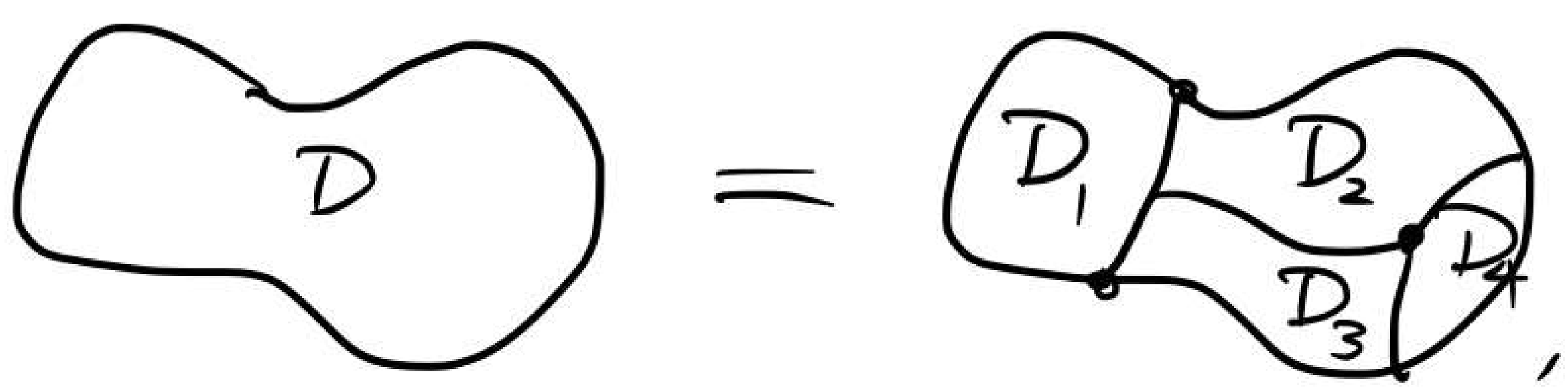
connected



not connected

③ Decomposing the domain.

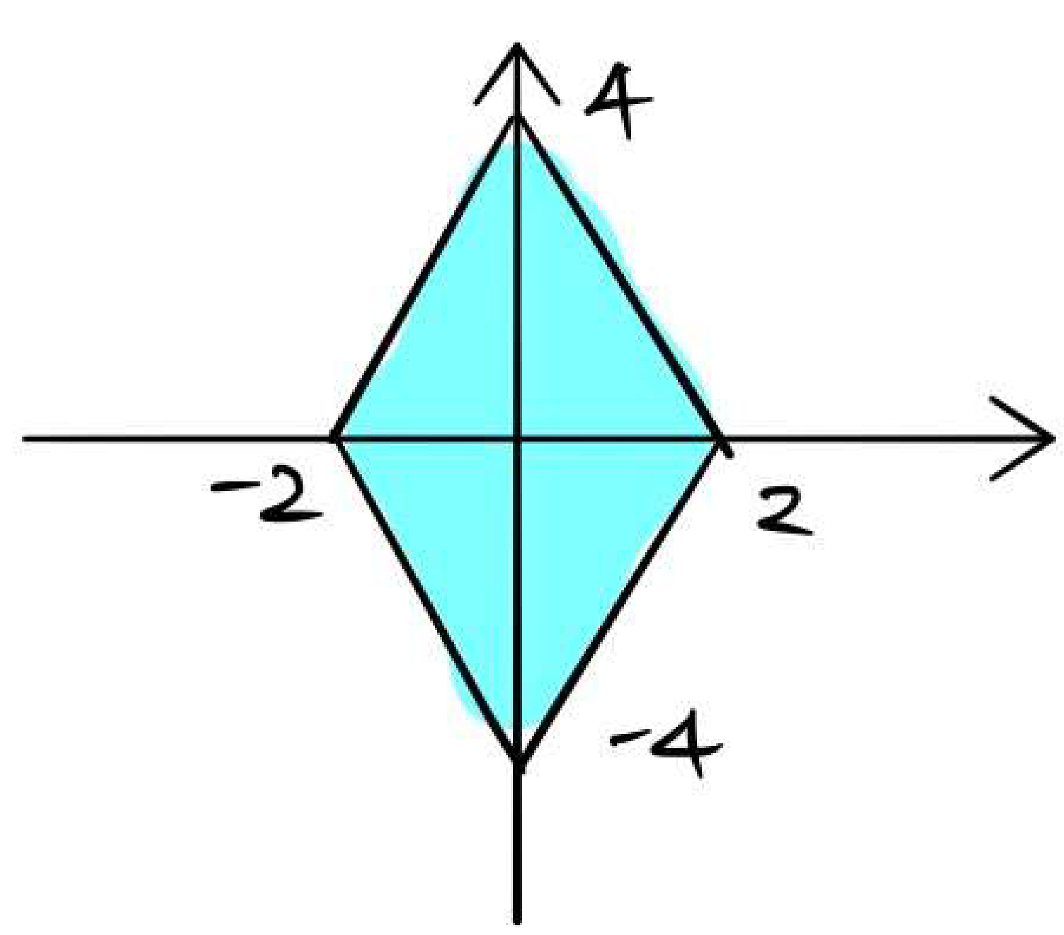
$\boxed{\text{Fact}}$ If $D = D_1 \cup \cdots \cup D_N$ where D_i's do not overlap except possibly on boundary,



then

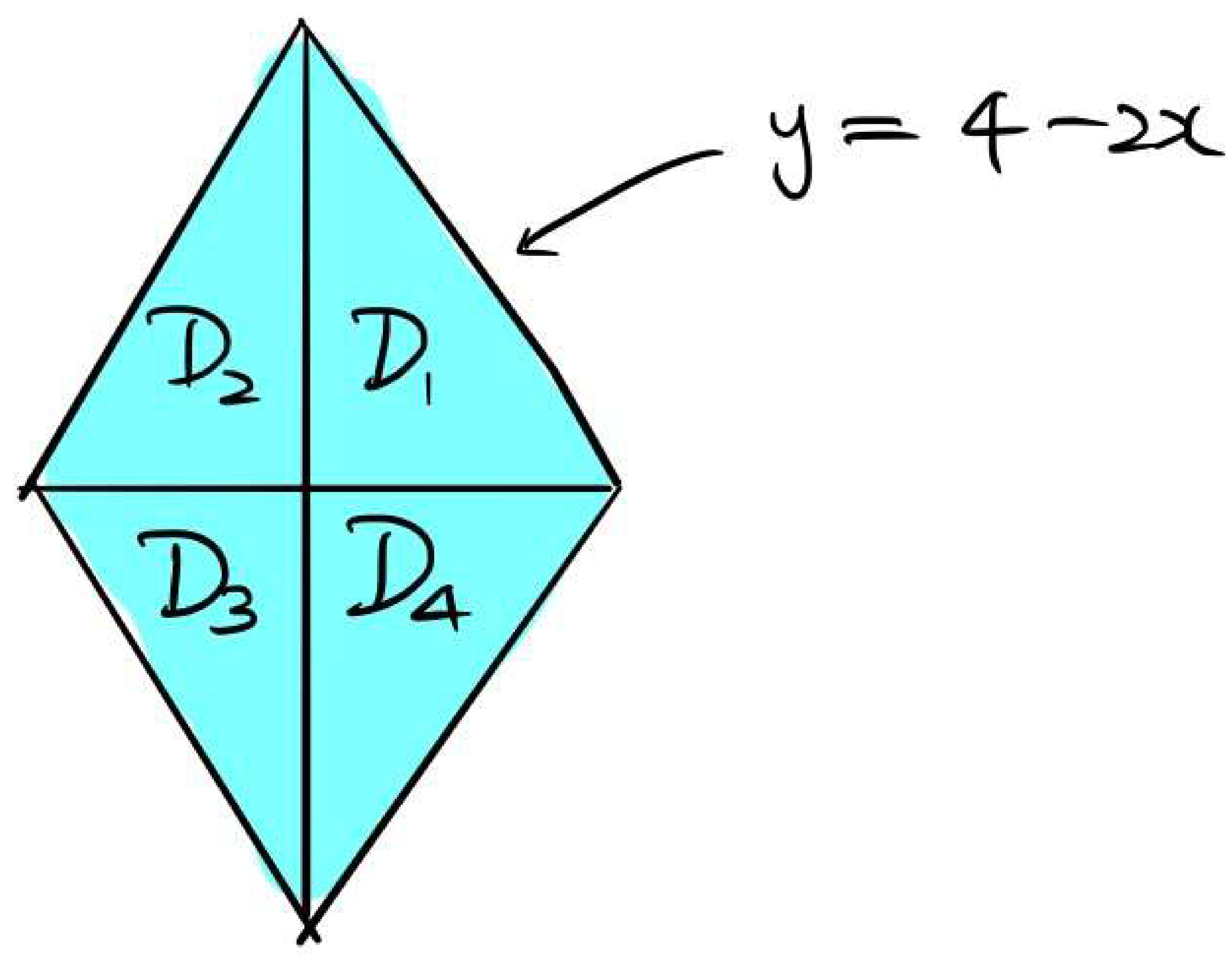
$$\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \cdots + \iint_{D_N} f(x,y)\, dA$$

$\boxed{\text{Ex}}$ Let $D = \{(x,y) : 2|x| + |y| \le 4\}$.



Compute $\iint_D y^2\,dA$.

<u>Sol)</u> Decompose D as



Then by $\begin{cases} \text{symmetry of the domain \&} \\ \text{symmetry of the function } f(x,y) = y^2 \end{cases}$

$f(x,y) = f(-x,y) = f(x,-y) = f(-x,-y)$,

we get:

$$\iint_{D_i} f(x,y)\,dA = \iint_{\mathbf{D_1}} f(x,y)\,dA.$$

So

$$\begin{aligned} \iint_D y^2\,dA &= \sum_{i=1}^{4} \iint_{D_i} y^2\,dA \\ &= 4\iint_{D_1} y^2\,dA \\ &= 4\int_0^2 \int_0^{4-2x} y^2\,dy\,dx \\ &= 4\int_0^2 \left[\frac{1}{3}y^3\right]_{y=0}^{4-2x} dx \\ &= \cdots \end{aligned}$$