HW: 1, 8, 9, 16, 17, 19, 23, 31, 40, 42, 49 in Section 16.1

OH: W 1–2:30
R 4–5:30
MS 6322

Note 2

Q We have defined double integral. But how to calculate this?

- We have seen geometric argument, arguement by symmetry, etc.
- We will learn a powerful technique using Fubini's theorom.

③ Iterated Integral

- iterated integral : several integrals, nested.
- $$\int_a^b \left(\int_c^d f(x,y)\, dy \right) dx.$$

  This integral is evaluated first by fixing x and computing the single-variable integral
  $$S(x) := \int_c^d f(x,y)\, dy$$
  and then computing
  $$\int_a^b S(x)\, dx.$$
  (I.e., the inner integral is computed first.)

- We usually omit the brackets for convenience:
  $$\int_a^b \int_c^d f(x,y)\, dy dx = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx.$$

- Similarly,

$$\int_c^d \int_a^b f(x,y)\, dx\, dy$$

evaluates $\int_a^b \cdots dx$ first by fixing y, and then $\int_c^d \cdots dy$.

$\boxed{\text{Ex}}$

$$\int_0^1 \int_2^3 (x^2 y + 3x)\, dy\, dx$$

$$= \int_0^1 \left[x^2 \cdot \frac{y^2}{2} + 3x \cdot y \right]_{y=2}^{3} dx$$

clarify which variable is being substituted.

$$= \int_0^1 \left(\left(\frac{9}{2}x^2 + 9x \right) - \left(2x^2 + 6x \right) \right) dx$$

$$= \int_0^1 \left(\frac{5}{2}x^2 + 3x \right) dx$$

$$= \left[\frac{5}{6}x^3 + \frac{3}{2}x \right]_0^1 = \frac{5}{6} + \frac{3}{2} = \frac{7}{3}.$$

$\boxed{\text{Ex}}$

$$\int_0^1 \int_0^2 e^{x+2y}\, dx\, dy$$

$$= \int_0^1 \left[e^{x+2y} \right]_{x=0}^{2} dy$$

$$= \int_0^1 \left(e^{2+2y} - e^{2y} \right) dy$$

$$= \left[\frac{1}{2} e^{2+2y} - \frac{1}{2} e^{2y} \right]_{y=0}^{1}$$

$$= \left(\frac{1}{2} e^{2+2} - \frac{1}{2} e^{2} \right) - \left(\frac{1}{2} e^{2} - \frac{1}{2} \right) = \frac{1}{2} e^4 - e^2.$$

$\boxed{\text{THM}}$ (Fubini's Theorem) If $R = [a,b]\times[c,d]$ and $f: R \to \mathbb{R}$ is continuous, then

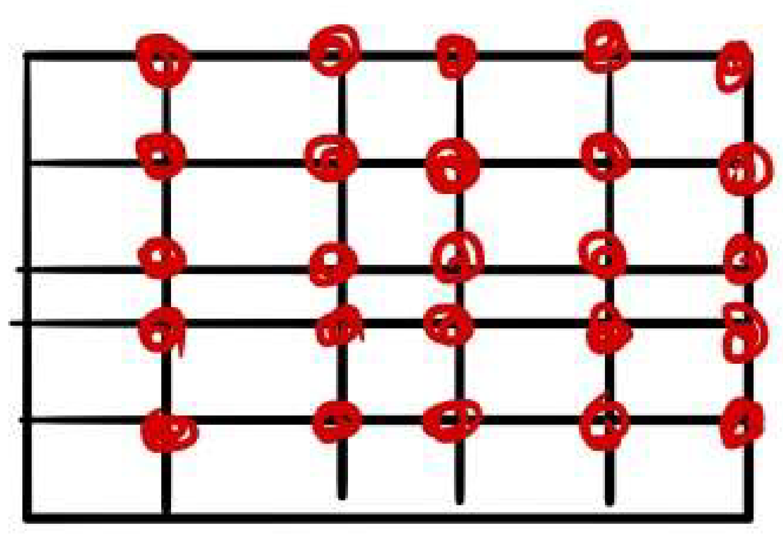
$$\int_a^b \int_c^d f(x,y)\,dy\,dx$$
$$= \iint_R f(x,y)\,dA$$
$$= \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

Sketch of pf)

- We know that f is integrable on R. Recall that this means:

$$\text{Limit} \quad \iint_R f(x,y)\,dA := \lim_{\|P\|\to 0} \sum_{i=1}^{N}\sum_{j=1}^{M} f(P_{ij})\,\Delta x_i \Delta y_j \quad \text{exists.}$$

- May compute this limit along partitions with $P_{ij} = (x_i, y_j)$, i.e.,

$P =$  .

P_{ij} = top-right corner of the subrectangle R_{ij}.

- Then

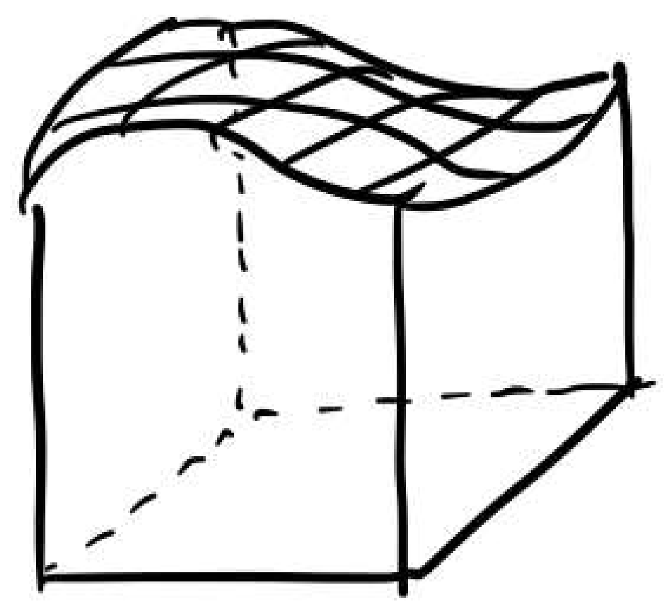
$$\iint_R f(x,y) = \lim_{\substack{N\to\infty \\ M\to\infty}} \boxed{\sum_{i=1}^{N} \boxed{\sum_{j=1}^{M} f(x_i, y_j)\,\Delta y_j}\,\Delta x_i}$$

$$\approx \int_c^d f(x_i, y)\,dy = S(x_i)$$

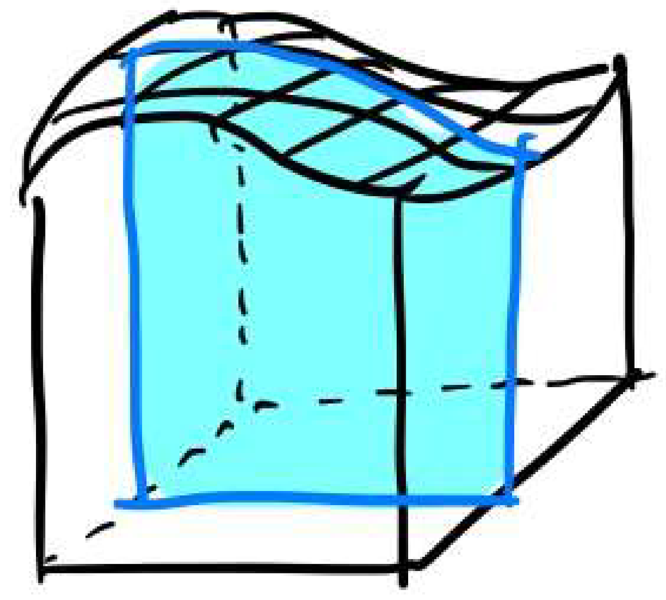
$$\approx \int_a^b S(x)\,dx = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

- Graphically,

$$\iint_R f(x,y)\, dA$$

$$= \Big[\text{ signed vol. of }$$  $$\Big]$$

$$= \int_a^b \Big[\text{ signed area of the cross-section }$$



$$\text{in the plane } x = x_0 \Big] \, dx_0.$$

$$= \int_a^b \int_c^d f(x_0, y)\, dy\, dx_0.$$

- Switching the order of summations proves the other half. ∎

$\boxed{\text{Consequences}}$ Let $R = [a,b] \times [c,d]$ and $f : R \to \mathbb{R}$ conti.

1) Can interchange the order of integration:

$$\int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

2) If f is of the form $f(x,y) = g(x)h(y)$, then

$$\iint_R f(x,y)\, dA = \left(\int_a^b g(x)\, dx \right) \left(\int_c^d h(y)\, dy \right).$$

Indeed,

does not depend on y, hence acts like a const. for $\int_c^d \ldots dy$.

$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_c^d g(x)\, h(y)\, dy \right) dx$$

$$= \int_a^b g(x) \boxed{\left(\int_c^d h(y)\, dy \right)} dx$$

constant indep. of x.

$$= \left(\int_c^d h(y)\, dy \right)\left(\int_a^b g(x)\, dx \right)$$

$\boxed{\text{Ex}}$ $\iint_R (x^2y + 3x)\, dy\,dx$, $\quad R = [0,1]\times[2,3]$

$$= \iint_R x^2 y\, dy\,dx + \iint_R 3x\, dy\,dx$$

$$= \left(\int_0^1 x^2\,dx \right)\left(\int_2^3 y\,dy \right) + \left(\int_0^1 3x\,dx \right)\left(\int_2^3 1\,dy \right)$$

$$= \frac{1^3 - 0^3}{3} \cdot \frac{3^2 - 2^2}{2} + \frac{3}{2}(1^2 - 0^2)\cdot(3-2)$$

$$= \frac{5}{6} + \frac{3}{2} = \frac{7}{3}.$$

- <u>Q</u> When applying Fubini's thm, which order of integration to use?

<u>A</u> Use whichever works better.
(In many cases, it doesn't matter to much. But in some cases, one is easier than the other.)

$\boxed{\text{Ex}}$ Let $R = [0,1] \times [0,1]$. Evaluate

$$I := \iint_R x e^{xy}\, dA.$$

<u>Sol)</u> • $I := \int_0^1 \int_0^1 x e^{xy}\, dx\,dy$ by Fubini.

Using $\int x e^x\, dx = (x-1)e^x + C$ and the sub $u = xy$,

$$\int_0^1 x e^{xy}\, dx = \frac{1}{y^2} \int_0^y u e^u\, du = \frac{(y-1)e^y + 1}{y^2}.$$

So

$$I = \int_0^1 \frac{(y-1)e^y + 1}{y^2}\, dy = \cdots ??$$

• Instead, use the order

$$I := \int_0^1 \int_0^1 x e^{xy}\, dy\,dx.$$

Using the sub $u = xy$,

$$= \int_0^1 \int_0^x e^u\, du\,dx$$

$$= \int_0^1 (e^x - 1)\, dx$$

$$= e - 2.$$