Lecture 1

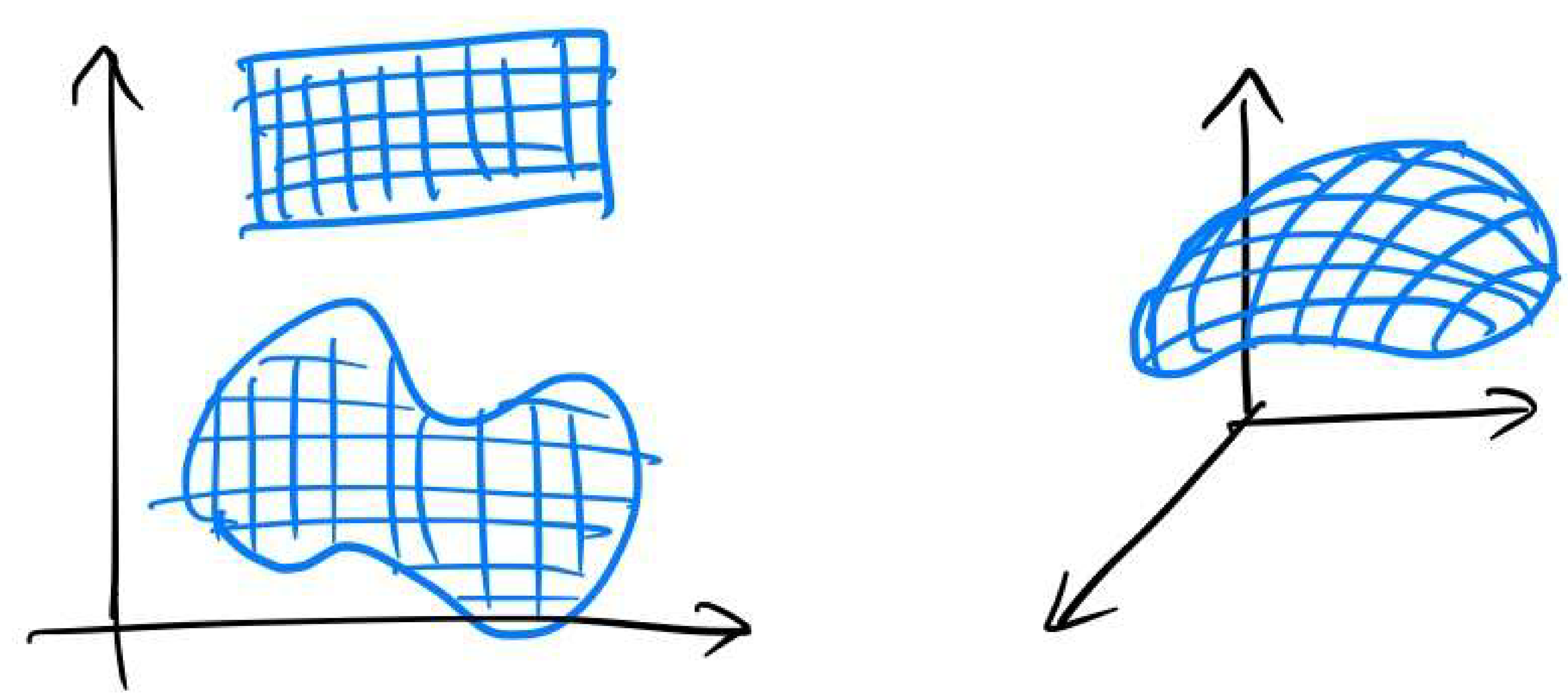
GOAL Learn how to define & compute integrals of functions on "geometric spaces".

① Function

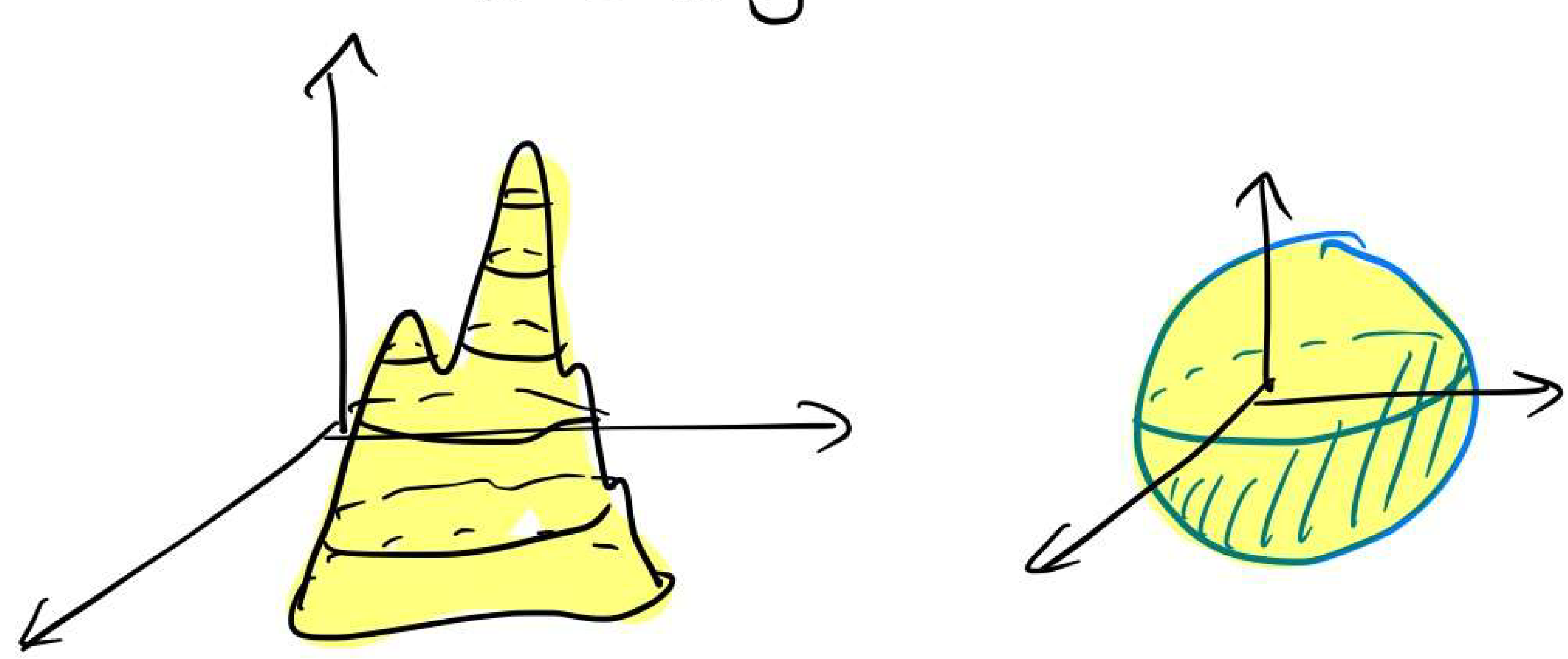
- Geometric spaces :

line/ curve

planar domain/ surface



solid body in $\mathbb{R}^3$



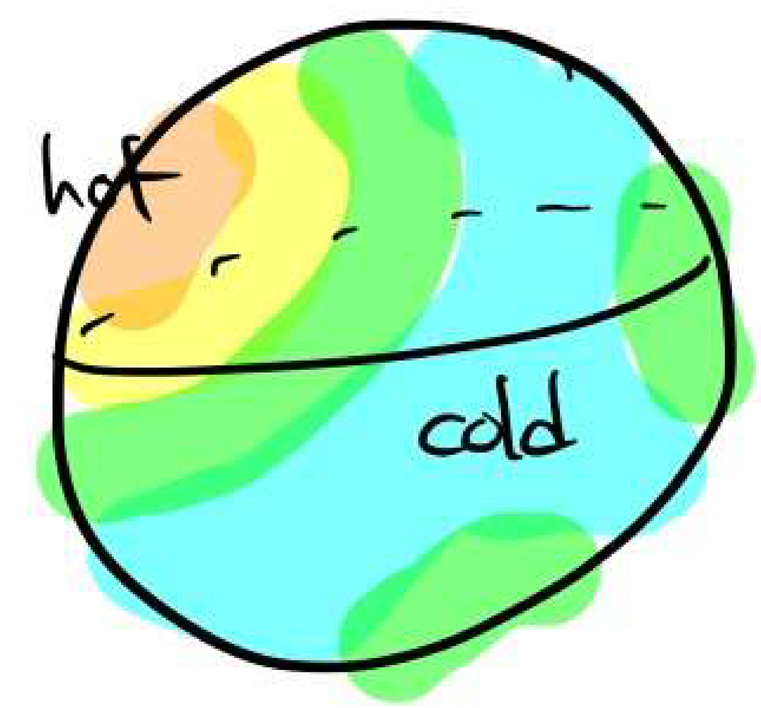
- In this class, will concentrate on functions which assign, to each point in a geometric space,

number / vector

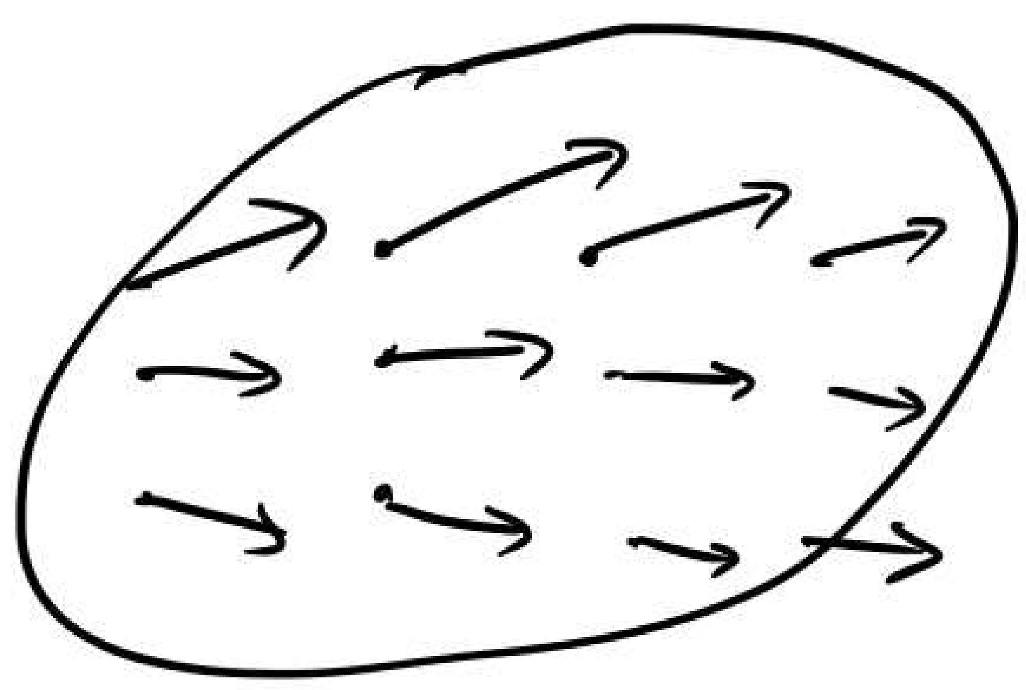
(scalar field) (vector field)

Ex ▷ density map of a wire
(scalar field on a curve)

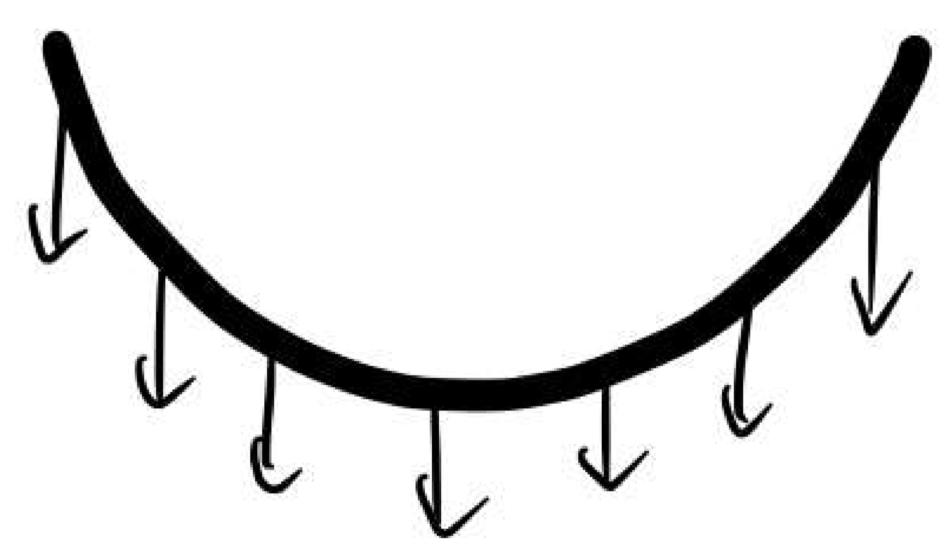
▷ temperatures map of the Earth
(scalar field on a surface)



▷ wind velocity
(vector field)



▷ gravity on a hanged wire
(vector field on a curve)



② Integration in two variables

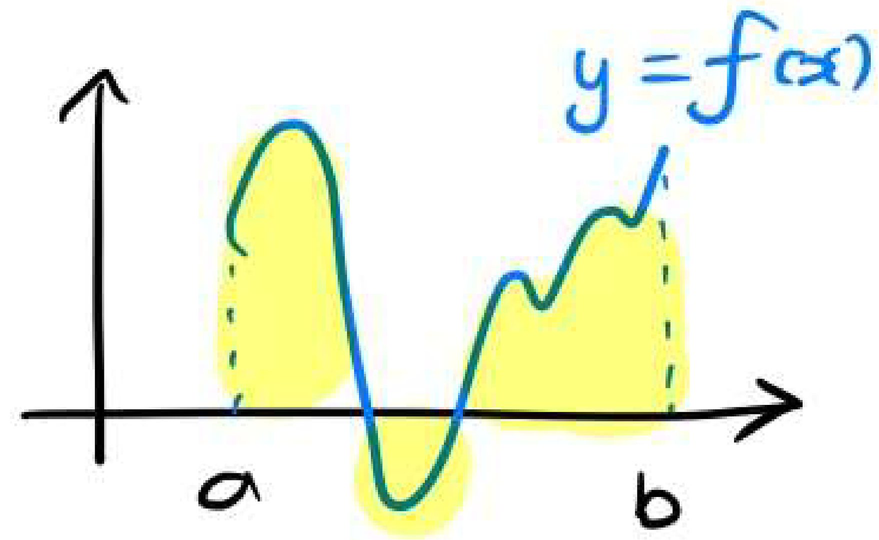
- Recall

$$\left[\text{definite integral } \int_a^b f(x)\,dx\right]$$

$$= \left[\text{limit of Riemann sum } \sum_{i=1}^{N} f(x_i)\,\Delta x_i\right]$$

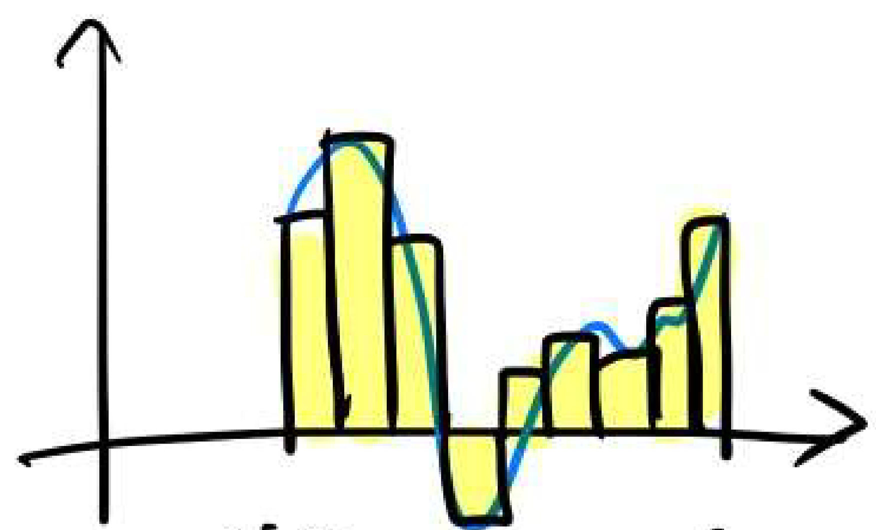
$$= \left[\begin{array}{l}\text{limit of the sum of small quantities } f(x_i)\Delta x_i \\ \text{arising from partitioning}\end{array}\right]$$

Ex [signed area of



]

= [limit of sum of signed areas

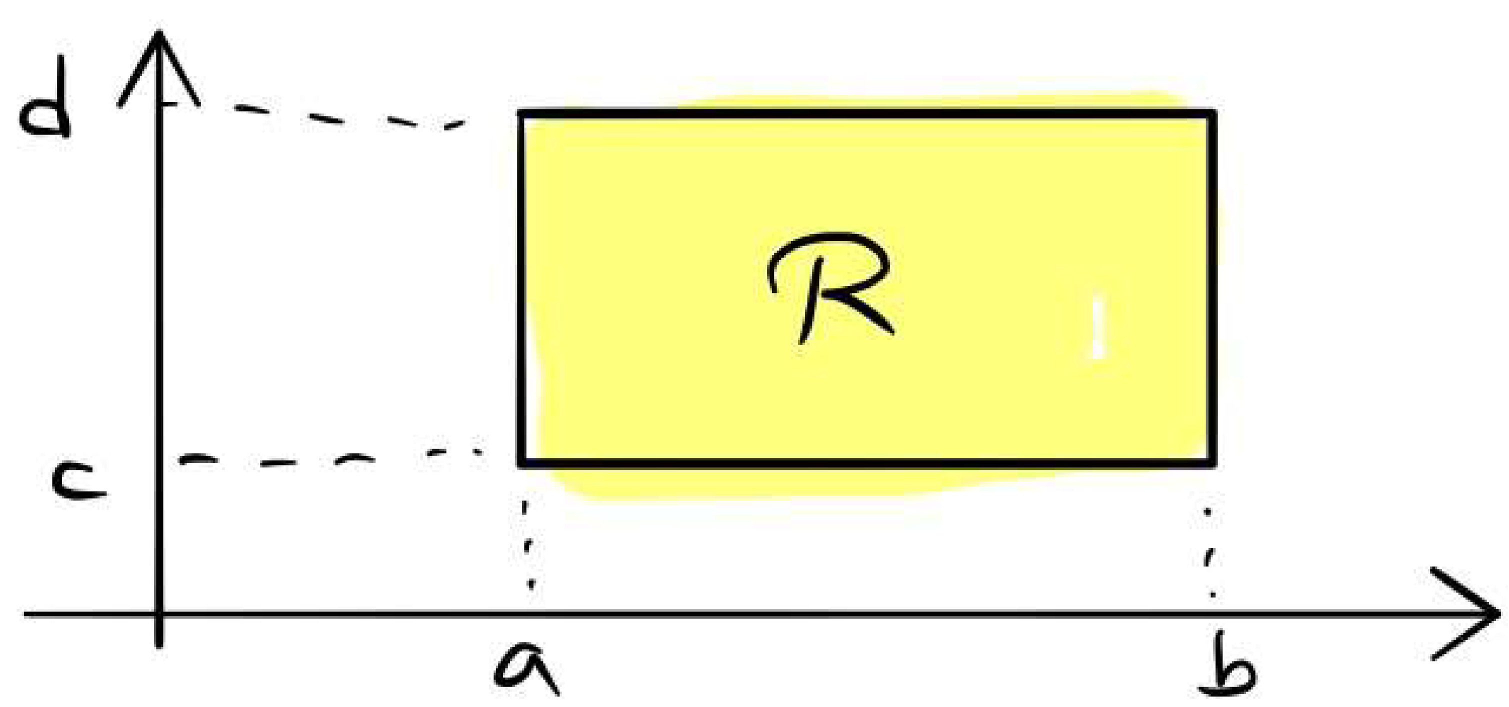


as the partition gets finer]

- Same idea can be applied to functions in two variables.

- We first focus on a function $f : R \to \mathbb{R}$ on a rectangle

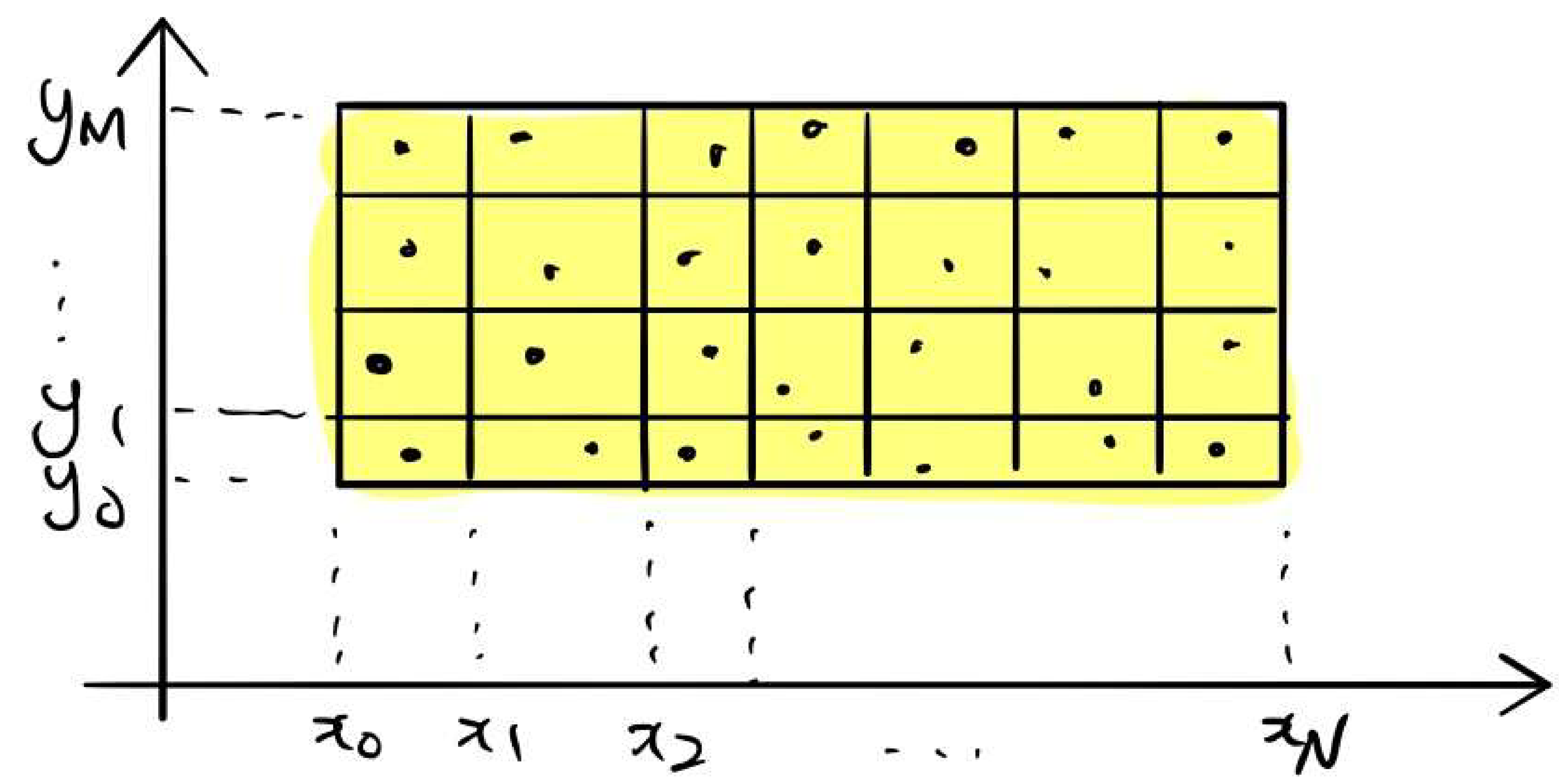
$$\begin{aligned} R &= [a,b] \times [c,d] \\ &= \{ (x,y) : a \leq x \leq b, \quad c \leq y \leq d \} \end{aligned}$$



- Double integral $\iint_R f(x,y)\, dA$ will be defined in the following steps:

(Step 1) Subdivide R by choosing partitions:

$$a = x_0 < x_1 < \cdots < x_N = b$$
$$c = y_0 < y_1 < \cdots < y_M = d$$



For each subrectangle $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, choose a sample point $P_{ij} \in R_{ij}$.

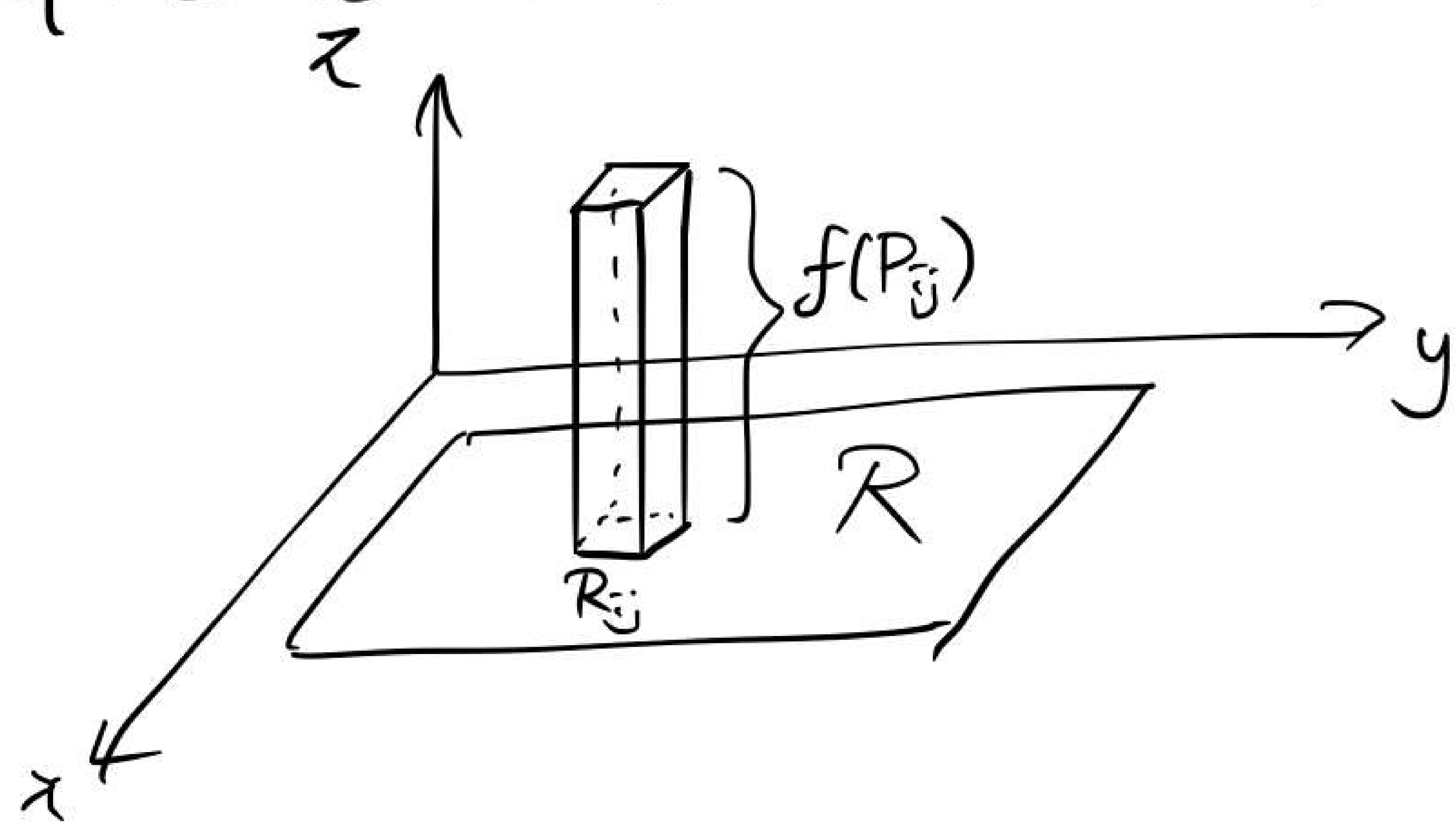
(Step 2) Sum the quantities

$$f(P_{ij}) \Delta A_{ij}, \qquad \Delta A_{ij} := [\text{area of } R_{ij}]$$

to form the Riemann sum.

$$S_{N,M} = \sum_{i=1}^{N} \sum_{j=1}^{M} f(P_{ij}) \Delta A_{ij}$$

$\boxed{\text{Ex}}$ If f represents the height, then $f(P_{ij}) \Delta A_{ij}$ represents the volume of the narrow box



(Step 3) Pass to the limit:

$\boxed{\text{Def}}$ The double integral of f on R is defined as

$$\iint_R f(x,y)\, dA := \lim_{\|P\|\to 0} S_{N,M} \qquad (*)$$

$$= \lim_{\|P\|\to 0} \sum_{i=1}^{N} \sum_{j=1}^{M} f(P_{ij}^*)\, \Delta A_{ij},$$

where

$$\|P\| = \max\{\Delta x_1, \cdots, \Delta x_N, \Delta y_1, \cdots, \Delta y_M\}.$$

If the limit $(*)$ exists, we say that f is integrable on R.

$\boxed{\text{THMs}}$ Let R be a rectangle.

(1) continuous functions on R is integrable.

(2) double integral on R is linear, i.e.,

$$\iint_R (f(x,y)+g(x,y))\, dA = \iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA,$$

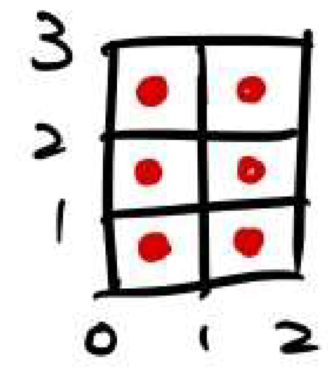
$$\iint_R cf(x,y)\, dA = c\iint_R f(x,y)\, dA$$

for integrable f, g and constant c.

(3) $$\iint_R 1\, dA = [\text{area of } R].$$

$\boxed{\text{Ex}}$ (Riemann sum) Consider

$$\begin{cases} R = [0,2] \times [0,3] \\ f(x,y) = x^2 - y^2 \end{cases}$$

$P =$ 

$$\begin{pmatrix} x_i = i, \quad y_j = j, \\ P_{ij} = (i-\frac{1}{2}, j-\frac{1}{2}) \end{pmatrix}$$

Then

$$\Delta A_{ij} = [\text{area of } R_{ij}] = 1,$$

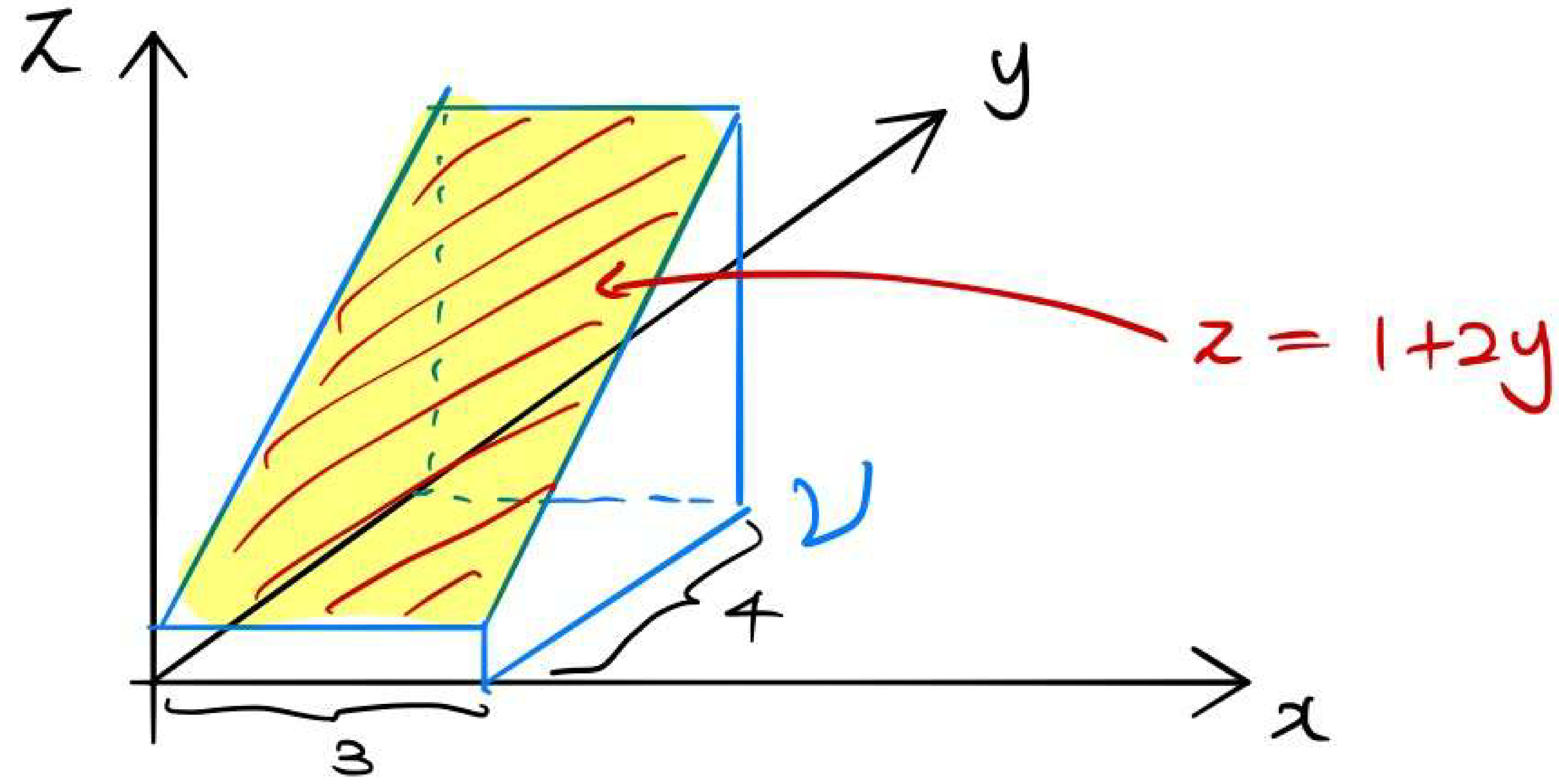
and so, the Riemann sum is

$$\begin{aligned} S_{2,3} &= \sum_{i=1}^{2} \sum_{j=1}^{3} f(P_{ij}) \Delta A_{ij} \\ &= f(\tfrac{1}{2},\tfrac{1}{2}) \cdot 1 + f(\tfrac{1}{2},\tfrac{3}{2}) \cdot 1 + f(\tfrac{1}{2},\tfrac{5}{2}) \cdot 1 \\ &\quad + f(\tfrac{3}{2},\tfrac{1}{2}) \cdot 1 + f(\tfrac{3}{2},\tfrac{3}{2}) \cdot 1 + f(\tfrac{3}{2},\tfrac{5}{2}) \cdot 1 \\ &= -10. \end{aligned}$$

$\boxed{\text{Ex}}$ (Using geometry) Let $R = [0,3] \times [0,4]$. Then

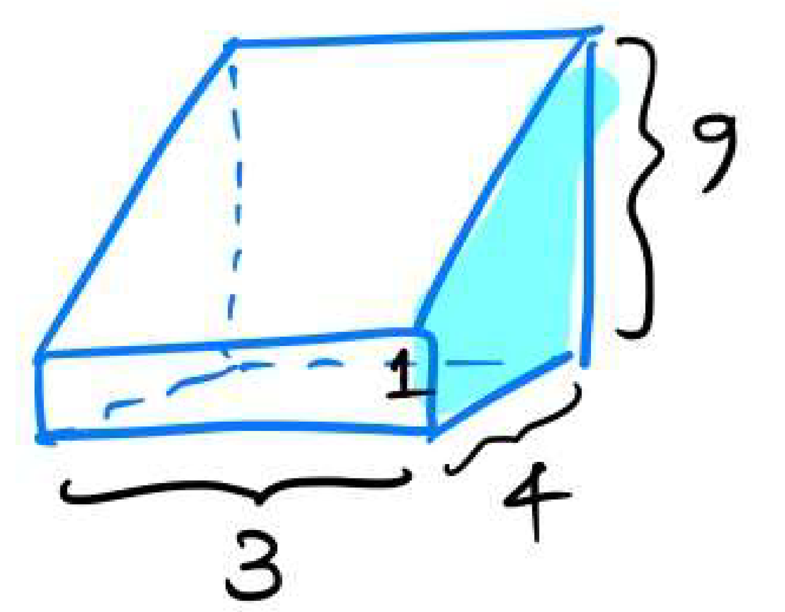
$$\iint_R (1+2y)\, dA$$

represents the (signed) volume of the solid V:

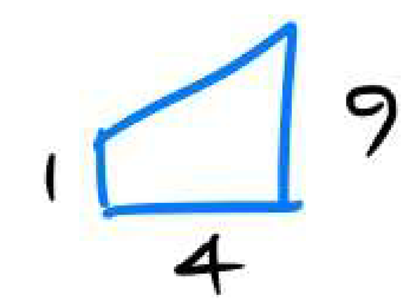


bounded between $z = 1+2y$ and $z=0$ on R.
So

$$\iint_R (1+2y)\, dA$$

$= \Big[$ volume of v = 

$\Big]$

$= 3 \cdot \Big[$ area of the side 

$\Big]$

$$= 3 \cdot \left(4 \cdot \frac{1+9}{2}\right)$$

$$= 60.$$

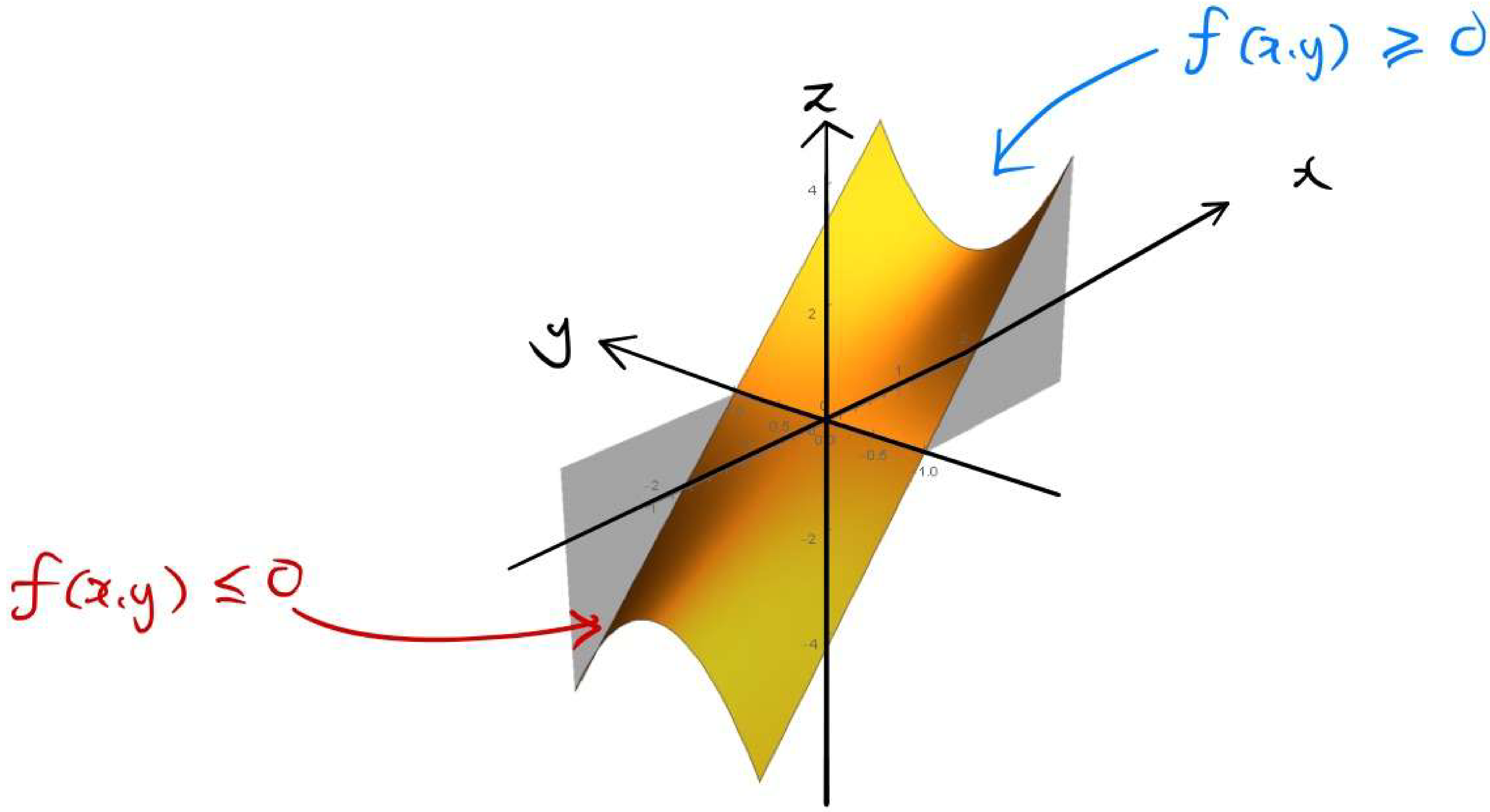
$\boxed{\text{Ex}}$ (Arguing by symmetry) Let $R = [-2,2] \times [-1,1]$ and $f(x,y) = x(1+y^2)$. Show

$$\iint_R f(x,y)\, dA = 0.$$

Sol) The region between

$$\begin{cases} z = x(1+y^2) \\ z = 0 \end{cases}$$

on R consists of two solids:



These two solids are congruent by the symmetry

$$f(-x, y) = -f(x,y).$$

So the signed volumes of two solids cancel out and the net signed volume of the region is 0. ////