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- Exercises are taken from J. Rogawski, C. Adams, R. Franzosa *Calculus, Multivariable*, 4th Ed., W. H. Freeman & Company.
- The questions marked with a star ($\star$) are either more difficult or of the form that is not intended for an exam. Nonetheless, they are worth thinking about.

18.1 Green's Theorem

1. Let $\mathcal{C}_R$ be the circle of radius R centered at the origin. Use the Green's Theorem to determine

$$\oint_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{r},$$

where $\mathbf{F}$ is a vector field such that

$$\oint_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = 6$$

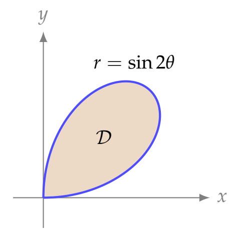
and

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = x^2 + y^2 \quad \text{on} \quad 1 \leq x^2 + y^2 \leq 9.$$

2. Use

$$\text{area}(\mathcal{D}) = \frac{1}{2} \oint_{\partial \mathcal{D}} x\, dy - y\, dx$$

to compute the area of the region $\mathcal{D}$ bounded by the curve in polar coordinates:



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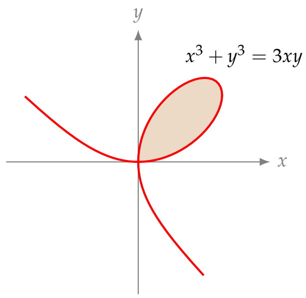
3. Let $x^3 + y^3 = 3xy$ be the **folium of Descartes**.



(a) Show that the loop of the folium has a parametrization in terms of $t = y/x$ given by

$$x = \frac{3t}{1+t^3}, \qquad y = \frac{3t^2}{1+t^3}, \qquad (0 \le t < \infty)$$

(b) Show that

$$x\,\mathrm{d}y - y\,\mathrm{d}x = \frac{9t^2}{(1+t^3)^2}\,\mathrm{d}t.$$

Hint: By the Quotient Rule, we have $x^2\,\mathrm{d}(y/x) = x\,\mathrm{d}y - y\,\mathrm{d}x$.

(c) Find the area of the loop of the folium.

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4. Area of a Polygon Green's Theorem leads to a convenient formula for the area of a polygon.

(a) Let $\mathcal{C}$ be the line segment joining (x_1, y_1) to (x_2, y_2). Show that

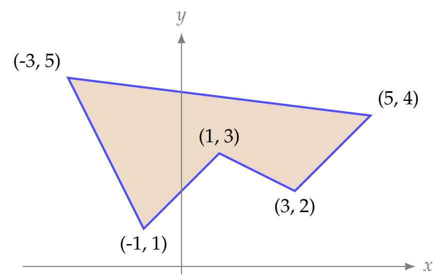
$$\frac{1}{2}\int_{\mathcal{C}} x\,\mathrm{d}y - y\,\mathrm{d}x = \frac{1}{2}(x_1y_2 - x_2y_1).$$

(b) Prove that the area of the polygon with vertices (x_1, y_1), $(x_2, y_2), \ldots, (x_n, y_n)$ is equal to

$$\frac{1}{2}\sum_{i=1}^{n}(x_iy_{i+1} - x_{i+1}y_i),$$

where we set $(x_{n+1}, y_{n+1}) = (x_1, y_1)$.

(c) Use the result of the previous part to compute the areas of the polygons in the following figure:



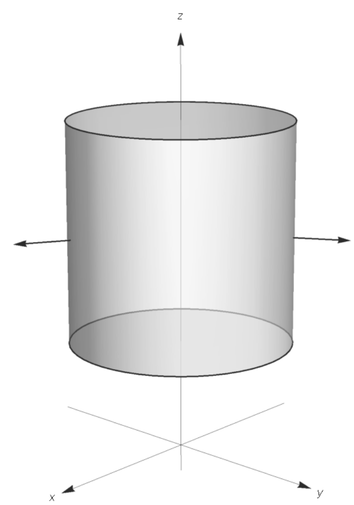
5. Compute the flux

$$\oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}s$$

of $\mathbf{F}(x, y) = \langle 2x + y^3, 3y - x^4 \rangle$ across the unit circle $\mathcal{C}$ oriented counter-clockwise, using the Flux Form of Green's Theorem.

18.2 Stokes' Theorem

6. Let $\mathcal{S}$ be the surface of the cylinder (not including the top and bottom) of radius 2 for $1 \leq z \leq 6$, oriented with outward-pointing normal.



(a) Indicate with arrow the orientation of $\partial \mathcal{S}$ (the top and bottom circles).

(b) Verify Stokes' Theorem for $\mathcal{S}$ and $\mathbf{F} = \langle yz^2, 0, 0 \rangle$.

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In Exercises 7–9, calculate curl($\mathbf{F}$) *and then apply Stokes' Theorem to compute the flux of* curl($\mathbf{F}$) *through the given surface using a line integral.*

7. $\mathbf{F} = \langle e^{z^2} - y, e^{z^3} + x, \cos(xz) \rangle$, the upper half of the unit sphere $x^2 + y^2 + z^2 = 1,\ z \geq 0$ with outward-pointing normal.

8. $\mathbf{F} = \langle yz, -xz, z^3 \rangle$, that part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the two planes $z = 2$ and $z = 4$ with upward-pointing unit normal vector.

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9. $\mathbf{F} = \langle yz, xz, xy \rangle$, that part of the cylinder $x^2 + y^2 = 1$ that lies between the two planes $z = -1$ and $z = 2$ with outward-pointing unit normal vector.

In Exercises 10–11, apply Stokes' Theorem to evaluate $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ *by finding the flux of* $\operatorname{curl}(\mathbf{F})$ *across an appropriate surface*

10. $\mathbf{F} = \langle yz, xz, xy \rangle$, $\mathcal{C}$ is the square with vertices $(0,0,2)$, $(1,0,2)$, $(1,1,2)$, $(0,1,2)$, oriented counter-clockwise as viewed from above.

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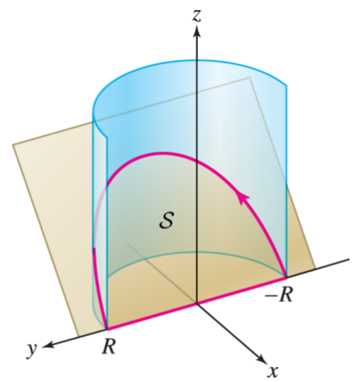
11. $\mathbf{F} = \langle y, -2z, 4x \rangle$, $\mathcal{C}$ is the boundary of that portion of the plane $x + 2y + 3z = 1$ that is in the first octant (i.e., $x, y, z \geq 0$) of the space, oriented counter-clockwise as viewed from above.

12. Let $\mathcal{S}$ be the portion of the plane $z = -x$ contained in the half-cylinder of radius R depicted in the following figure:



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Use Stokes' Theorem to calculate the circulation of

$$\mathbf{F} = \langle z, -x, y + 2z \rangle$$

around the boundary of $\mathcal{S}$ (a half-ellipse) in the counter-clockwise direction when viewed from above.

Hint: Show that $\mathrm{curl}(\mathbf{F})$ *is orthogonal to the normal vector to the plane.*

13. Let $\mathbf{F} = \langle 0, -z, 1 \rangle$. Let $\mathcal{S}$ be the spherical cap $x^2 + y^2 + z^2 = 1$, where $z \geq \frac{1}{2}$, with upward-pointing unit normal vector. Evaluate

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$$

directly as a surface integral. Then verify that $\mathbf{F} = \mathrm{curl}(\mathbf{A})$, where $\mathbf{A} = \langle 0, x, xz \rangle$ and evaluate the surface integral again using Stokes' Theorem.

14. Let **A** be the vector potential and **B** the magnetic field of the infinite solenoid of radius R in Example 4. Use Stokes' Theorem to compute:

(a) The flux of **B** through a surface whose boundary is a circle in the xy-plane of radius $r < R$.

(b) The circulation of **A** around the boundary $\mathcal{C}$ of a surface lying outside the solenoid.

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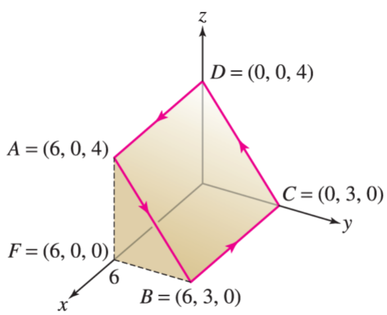
15. A uniform magnetic field $\mathbf{B}$ has constant strength b in the z-direction, i.e., $\mathbf{B} = \langle 0, 0, b \rangle$.

(a) Verify that $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$ is a vector potential for $\mathbf{B}$, where $\mathbf{r} = \langle x, y, 0 \rangle$.

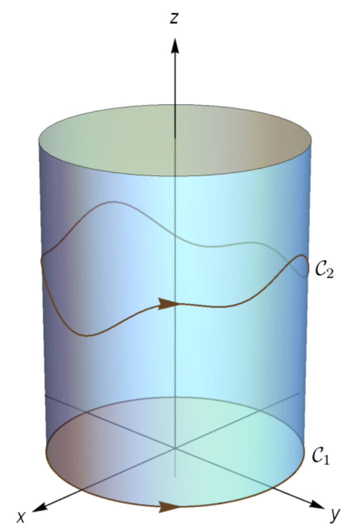
(b) Calculate the flux of $\mathbf{B}$ through the rectangle with vertices A, B, C and D in:



16. Let $\mathbf{F} = \langle -y, x, z^2 \rangle$. Let

- $\mathcal{C}_1$: the circle $x^2 + y^2 = R^2$, $z = 0$, and
- $\mathcal{C}_2$: any closed curve going around the cylinder $x^2 + y^2 = R^2$.

Suppose that both $\mathcal{C}_1$ and $\mathcal{C}_2$ are oriented counter-clockwise as viewed from above, as in the figure:



Compute

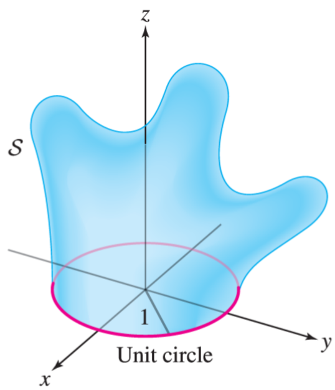
$$\oint_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}.$$

Hint: Show that $\oint_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} = \oint_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r}$ *using the Stokes' Theorem.*

17. Suppose you know two things about a vector field $\mathbf{F}$:

- $\mathbf{F}$ has a vector potential $\mathbf{A}$ (but $\mathbf{A}$ is unknown).
- The circulation of $\mathbf{A}$ around the unit circle $x^2 + y^2 = 1, \; z = 0$ (oriented counter-clockwise) is 25.

Determine the flux of $\mathbf{F}$ through the surface $\mathcal{S}$ in the following figure, oriented with an "upward-pointing normal"[1].



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[1] The usage of this term is technically incorret because the unit normal vector $\mathbf{n}$ for the surface $\mathcal{S}$ in the figure can point both the positve and negative z-direction. To make the problem precise, $\mathcal{S}$ is oriented in such a way that $\partial\mathcal{S}$ with the boundary orientation is oriented counter-clockwise when viewed from above.