Math 32B Lecture 2, Winter 2020	Homework 6	Due February 14, in class
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- Exercises are taken from J. Rogawski, C. Adams, R. Franzosa *Calculus, Multivariable*, 4th Ed., W. H. Freeman & Company.

17.2 Line Integrals

Preliminary questions in the textbook

1. What is the line integral of the constant function $f(x, y, z) = 10$ over a curve $\mathcal{C}$ of length 5?

2. Which of the following have a zero integral over the vertical segment from $(0, 0)$ to $(0, 1)$?

(a) $f(x, y) = x$ **(b)** $f(x, y) = y$
(c) $\mathbf{F}(x, y) = \langle x, 0\rangle$ **(d)** $\mathbf{F}(x, y) = \langle y, 0\rangle$
(e) $\mathbf{F}(x, y) = \langle 0, x\rangle$ **(f)** $\mathbf{F}(x, y) = \langle 0, y\rangle$

Note: There may be multiple answers.

4. Suppose that $\mathcal{C}$ has length 5. What is the value of

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

in each of the following cases?

(a) $\mathbf{F}(P)$ is normal to $\mathcal{C}$ at all points P on $\mathcal{C}$.
(b) $\mathbf{F}(P)$ is a unit vector pointing in the negative direction along the curve.

Exercises outside the textbook

1. Let $\mathcal{C}$ be the line segment joining $(0, 0, 0)$ to $(3, 2, 1)$, and let $f(x, y, z) = 2y + xz$. In each of the following, evaluate

$$\int_{\mathcal{C}} f(x, y, z)\, ds$$

using the parametrization specified.

(a) $\mathbf{r}(t) = \langle 3t, 2t, t\rangle$ for $0 \le t \le 1$.

(b) $\mathbf{r}(t) = \langle 3\cos t, 2\cos t, \cos t\rangle$ for $0 \le t \le \frac{\pi}{2}$.

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2. Let $\mathcal{C}$ be the curve $y = x^{-1/2}$ for $1 \le x \le 2$, oriented from left to right, and let $\mathbf{F}(x,y) = \langle x(y+1), 2x^2 \rangle$. In each of the following, evaluate

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

using the parametrization specified.

(a) $\mathbf{r}(t) = \langle t, t^{-1/2} \rangle$ for $1 \le t \le 2$.

(b) $\mathbf{r}(t) = \langle e^t, e^{-t/2} \rangle$ for $0 \le t \le \ln 2$.

3. Integrate $f(x,y) = y\sqrt{2+x}$ over the upper semicircle $x^2 + y^2 = 1,\ y \ge 0$.

4. Integrate $f(x,y,z) = ye^{z^2}$ over the piecewise linear path from $(0,0,1)$ to $(2,0,0)$ to $(0,1,1)$.

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5. Calculate $\int_{\mathcal{C}} 1\, \mathrm{d}s$, where the curve $\mathcal{C}$ is parametrized by $\mathbf{r}(t) = \langle 1+2t, 3-t, 4+2t\rangle$ for $0 \le t \le 1$. What does this integral represent?

6. Integrate $\mathbf{F}(x,y) = \langle x^2, xy\rangle$ over the line segment from $(0,1)$ to $(2,-1)$.

7. Integrate $\mathbf{F}(x,y) = \langle xy, -2\rangle$ over $\frac{1}{4}x^2 + y^2 = 1$ with $x \ge 0$, $y \ge 0$.

8. Compute $\int_{\mathcal{C}} x^2\, \mathrm{d}x$, where $\mathcal{C}$ is the curve $y = x^3$ for $0 \le x \le 3$, oriented from left to right.

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9. Compute $\int_{\mathcal{C}} 3\,\mathrm{d}x + (x-y)\,\mathrm{d}y + z\,\mathrm{d}z$, where $\mathcal{C}$ is parametrized by $\mathbf{r}(t) = \langle 2+t^{-1}, t^2, t^2\rangle$ for $2 \le t \le 4$.

10. Compute the total mass of a metal wire loop

$$\mathbf{r}(t) = \langle \cos t, \sin t, \sin^2 t\rangle, \qquad 0 \le t \le 2\pi$$

in centimeters, assuming a mass density of

$$\rho(x,y,z) = \frac{3}{\sqrt{1+4x^2y^2}}\ \mathrm{g/cm}.$$

11. A charged semicircle

$$\mathbf{r}(t) = \langle \cos t, \sin t, 0\rangle, \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

(in meters) has charge density

$$\rho(x,y,0) = 10^{-8}(2-x)\ \mathrm{C/m}.$$

Find the electric potential at $P = (0,0,a)$.

12. Calculate the work done by $\mathbf{F} = \langle x, y, -10\rangle$ when the object moves along the path $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$ for $0 \le t \le 4\pi$.

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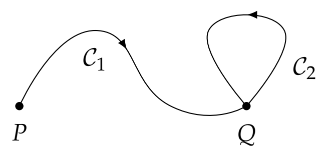
13. Let $\mathcal{C}_1$ be a path from P to Q, and $\mathcal{C}_2$ be an oriented loop at Q as below:



and $\mathbf{F}$ be a vector field such that

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = 3 \quad \text{and} \quad \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} = 6.$$

Determine:

(a) The integral of $\mathbf{F}$ over that path that traverses $\mathcal{C}_1$ in the opposite orientation, i.e.,

$$\int_{-\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r}$$

(b) The integral of $\mathbf{F}$ over the path obtained by concatenating $\mathcal{C}_1$ and $\mathcal{C}_2$, i.e.,

$$\int_{\mathcal{C}_1+\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$$

(c) The integral of $\mathbf{F}$ over the path that traverses $\mathcal{C}_1$ from P to Q and then back to P, i.e.,

$$\int_{\mathcal{C}_1-\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r}$$

(d) The integral of $\mathbf{F}$ over the path that traverses the loop $\mathcal{C}_2$ four times in the positive orientation of $\mathcal{C}_2$, i.e.,

$$\int_{4\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$$

14. Consider the *vortex field* $\mathbf{F}$ given by

$$\mathbf{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle.$$

Let $r(t)$ and $\theta(t)$ be real-valued functions on $[a,b]$ and let $\mathcal{C}$ be a curve avoiding the origin and parametrized by:

$$\mathbf{r}(t) = \langle r(t)\cos(\theta(t)), r(t)\sin(\theta(t)) \rangle, \qquad a \le t \le b.$$

Do the following:

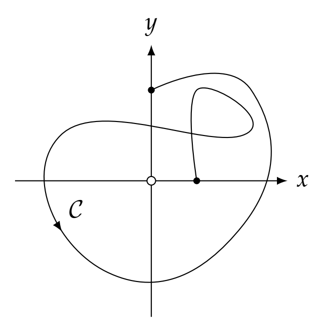
(a) Show that $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \theta'(t)$.

(b) Show that

$$\int_{\mathcal{C}} \frac{-y\,dx + x\,dy}{x^2+y^2} = \theta(b) - \theta(a),$$

i.e., the amount of rotation by the curve $\mathcal{C}$ about the origin in the counter-clockwise direction.

(c) Compute $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ for the curve $\mathcal{C}$ given by:



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17.3. Conservative Vector Fields

Exercises outside the textbook

1. Let $\mathbf{F}(x, y, z) = \nabla(x^2y + e^z)$, and let $\mathcal{C}$ be any path from $(0, 0, 0)$ to $(3, -1, 0)$. Evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

2. Let $\mathcal{C}$ be any path from (a, c) to (b, d). Compute

$$\int_{\mathcal{C}} x\,dx + 3y^2\,dy.$$

3. Consider the vector field

$$\mathbf{F}(x, y, z) = \langle yz^2, xz^2, 2xyz + 1 \rangle.$$

Its domain $\mathcal{D}$ is $\mathbf{R}^2$, which is simply connected. Do the following:

(a) Show that $\mathbf{F}$ is conservative by verifying the cross-partial condition.

(b) Compute $f(x, y, z) = \int_{\overline{OP}} \mathbf{F} \cdot d\mathbf{r}$, where $\overline{OP}$ is the line segment from the origin O to $P = (x, y, z)$.

(c) Verify that f computed in the previous step is indeed a potential function of $\mathbf{F}$.

(d) Find the value of $\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$, where $\mathcal{C}$ is parametrized by $\mathbf{r}(t) = \langle t^{2020}, e^{t(1-t)}, \sin(\pi t^{42}) \rangle$ for $0 \leq t \leq 1$.

In Exercises 4–7, find a potential function for $\mathbf{F}$ *or determine that* $\mathbf{F}$ *is not conservative.*

4. $\mathbf{F} = \langle x, y, z \rangle$.

5. $\mathbf{F} = \langle y, z, x \rangle$.

6. $\mathbf{F} = \langle y^3 + e^z, 3xy^2, xe^z \rangle$

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7. $\mathbf{F} = \langle (2z+3)e^x, \sin y, 2e^x \rangle$.

8. Evaluate

$$\oint_{\mathcal{C}} (1+x+y)\,\mathrm{d}x + \sin z\,\mathrm{d}y + (y+1)\cos z\,\mathrm{d}z$$

where $\mathcal{C}$ is the ellipse $9x^2 + 16y^2 = 25$, oriented clockwise.

Hint: Decompose the integral into $\oint_{\mathcal{C}} y\,\mathrm{d}x$ plus everything else. Can you identify an integral of a conservative vector field in this decomposition?

9. Show that $g(x,y) = \arctan(y/x)$ is a potential function of the vortex field

$$\mathbf{F}(x,y) = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

on the right-half plane $\mathcal{D} = \{(x,y) : x > 0\}$. Does this contradict the fact that $\mathbf{F}$ has no potential function on the xy-plane minus the origin?