Homework 4

- Homework 4 is due January 31 in class.
- Exercises are taken from J. Rogawski, C. Adams, R. Franzosa *Calculus, Multivariable*, 4th Ed., W. H. Freeman & Company.
- The starred problems will not be graded.

16.5 Applications of Multiple Integrals

4. Find the total population within a 4-km radius of the city center (located at the origin) assuming a population density of $\delta(x, y) = 2000(x^2 + y^2)^{-0.2}$ people per square kilometer.

6. Find the total mass of the solid region $\mathcal{W}$ defined by $x \geq 0$, $y \geq 0$, $x^2 + y^2 \leq 4$, and $x \leq z \leq 32 - x$ (in centimeters) assuming a mass density of $\delta(x, y, z) = 6y$ g/cm^3.

9. Assume that the density of the atmosphere as a function of altitude h (in kilometers) above sea level is $\delta(h) = ae^{-bh}$ kg/km^3, where $a = 1.225 \times 10^9$ and $b = 0.13$. Calculate the total mass of the atmosphere contained in the cone-shaped region $\sqrt{x^2 + y^2} \leq h \leq 3$.

18. Show that the centroid of the sector in Figure 13 has y-coordinate $\overline{y} = \left(\frac{2R}{3}\right)\left(\frac{\sin\alpha}{\alpha}\right)$.

21. Find the centroid of the "ice cream cone" region bounded, in spherical coordinates, by the cone $\phi = \pi/3$ and the sphere $\rho = 2$, assuming a density of $\delta(x, y, z) = 1$.
Hint: By the rotational symmetry of both the region and the density function around the z-axis, the centroid must lie on the z-axis.

Extra problems

N1. Find the centroid of the region bounded between the circles $x^2 + y^2 = 1$ and $(x - \frac{1}{2})^2 + y^2 = \frac{1}{2^2}$ as in Figure N1, assuming the density $\delta(x, y) = 1$.

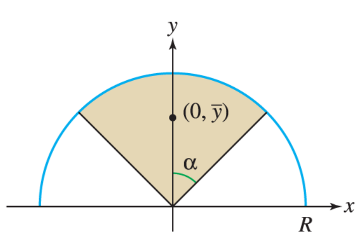


Figure 13

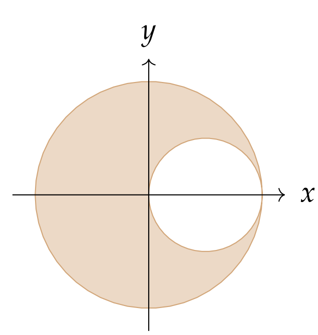


Figure N1

16.6 Change of Variables

Exercises

1.★ Determine the image under $G(u,v) = (2u, u+v)$ of the following sets:

(a) The u-axis and v-axis.
(b) The rectangle $\mathcal{R} = [0,5] \times [0,7]$.
(c) The line segment joining $(1,2)$ and $(5,3)$.
(d) The triangle with vertices $(0,1)$, $(1,0)$ and $(1,1)$.

2. Describe [in the form $y = f(x)$] the images of the lines $u = c$ and $v = c$ under the mapping $G(u,v) = (u/v, u^2 - v^2)$.

3. Let $G(u,v) = (u^2, v)$. Is one-to-one? If not, determine a domain on which G is one-to-one. Find the image under G of:

(a) The u-axis and $v - axis$.
(b) The rectangle $\mathcal{R} = [-1,1] \times [-1,1]$.
(c) The line segment joining $(0,0)$ and $(1,1)$.
(d) The triangle with vertices $(0,0)$, $(0,1)$, and $(1,1)$.

In Exercises 5–12, let $G(u,v) = (2u+v, 5u+3v)$ be a map from the uv-plane to the xy-plane.

7. Describe the image of the line $v = 4u$ under G in slope-intercept form.[1]

9. Show that the inverse of G is $G^{-1}(x,y) = (3x - y, -5x + 2y)$.
Hint. Show that $G(G^{-1}(x,y)) = (x,y)$ and $G^{-1}(G(u,v)) = (u,v)$.

11. Calculate $\text{Jac}(G) = \dfrac{\partial(x,y)}{\partial(u,v)}$.

12. Calculate $\text{Jac}(G^{-1}) = \dfrac{\partial(u,v)}{\partial(x,y)}$.

In Exercises 13–18, compute the Jacobian (at the point, if included).

13. $G(u,v) = (3u + 4v, u - 2v)$.

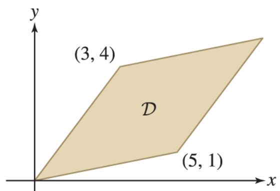
15. $G(r,t) = (r\sin t, r - \cos t), \quad (r,t) = (1, \pi)$.

16.★ $G(u,v) = (v \ln u, u^2 v^{-1}), \quad (u,v) = (1,2)$.

18. $G(u,v) = (ue^v, e^u)$.

21. Let $\mathcal{D}$ be the parallelogram in Figure 13. Apply the Change of Variables Formula to the map $G(u,v) = (5u + 3v, u + 4v)$ to evaluate $\iint_{\mathcal{D}} xy \, dx dy$ as an integral over $\mathcal{D}_0 = [0,1] \times [0,1]$.

[1] The equation of the form $y = mx + b$ for the slope m and the y-intercept b.



Rogawski et al., *Multivariable Calculus*, 4e, © 2019 W. H. Freeman and Company

Figure 13

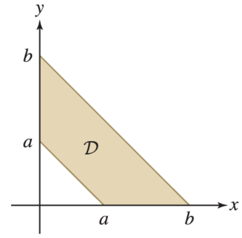


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Figure 14

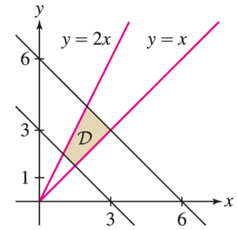

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Figure 17

22.* Let $G(u,v) = (u - uv, uv)$.

- **(a)** Show that the image of the horizontal line $v = c$ is $y = \frac{c}{1-c}x$ if $c \neq 1$ and is the y-axis if $c = 1$.
- **(b)** Determine the images of vertical lines in the uv-plane.
- **(c)** Compute the Jacobian of G.
- **(d)** Observe that by the formula for the area of a triangle, the region $\mathcal{D}$ in Figure 14 has area $\frac{1}{2}(b^2 - a^2)$. Compute this area again, using the Change of Variables Formula applied to G.
- **(e)** Calculate $\iint_{\mathcal{D}} xy \, \mathrm{d}x\mathrm{d}y$.

29. Let $\mathcal{D} = G(\mathcal{R})$, where $G(u,v) = (u^2, u+v)$ and $\mathcal{R} = [1,2] \times [0,6]$. Calculate $\iint_{\mathcal{D}} y \, \mathrm{d}x\mathrm{d}y$. *Note:* It is not necessary to describe

30.* Let $\mathcal{D}$ be the image of $\mathcal{R} = [1,4] \times [1,4]$ under the map $G(u,v) = (u^2/v, v^2/u)$.

- **(a)** Compute $\mathrm{Jac}(G)$.
- **(b)** Sketch $\mathcal{D}$.
- **(c)** Use the Change of Variables Formula to compute $\mathrm{area}(\mathcal{D})$ and $\iint_{\mathcal{D}} f(x,y) \, \mathrm{d}x\mathrm{d}y$, where $f(x,y) = x + y$.

32. Use the map $G(u,v) = \left(\frac{u}{v+1}, \frac{uv}{v+1}\right)$ to compute $\iint_{\mathcal{D}} (x+y) \, \mathrm{d}x\mathrm{d}y$, where $\mathcal{D}$ is the shaded region in Figure 17.

33. Show that $T(u,v) = (u^2 - v^2, 2uv)$ maps the triangle $T_0 = \{(u,v) : 0 \leq v \leq u \leq 1$ to the domain $\mathcal{D}$ bounded by $x = 0$, $y = 0$, and $y^2 = 4 - 4x$. Use T to evaluate $\iint_{\mathcal{D}} \sqrt{x^2 + y^2} \, \mathrm{d}x\mathrm{d}y$.

37.* Sketch the domain $\mathcal{D}$ bounded by $y = x^2$, $y = \frac{1}{2}x^2$, and $y = x$. Use a change of variables with the map $x = uv$, $y = u^2$ to calculate

$$\iint_{\mathcal{D}} y^{-1} \, \mathrm{d}x\mathrm{d}y.$$

This is an improper integral since $f(x,y) = y^{-1}$ is undefined at but it becomes proper after changing variables.

38. Find an appropriate change of variables to evaluate

$$\iint_{\mathcal{R}} (x+y)^2 e^{x^2-y^2}$$

where is the square with vertices $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$.

Extra problems

N1. Using the map $(x,y) = G(u,v)$ satisfying $x+y=u$ and $y=uv$, show that

$$\int_0^1 \int_0^{1-x} e^{\frac{y}{x+y}}\, \mathrm{d}y\mathrm{d}x = \frac{e-1}{2}.$$

Hint: This G is exactly the same as in Exercise 22 and the domain of integration $\mathcal{D}$ in the xy-plane corresponds to Figure 14 with $a=0$ and $b=1$.

N2. **(Symmetry argument revisited)** Let $\mathcal{D}_+$ be a region in the xy-plane lying entirely on and/or above the x-axis and let $\mathcal{D}_-$ be the reflection of $\mathcal{D}_+$ through the x-axis.

(a) Show that

$$\iint_{\mathcal{D}_+} f(x,y)\,\mathrm{d}x\mathrm{d}y = \iint_{\mathcal{D}_-} f(x,-y)\,\mathrm{d}x\mathrm{d}y.$$

Hint: Find the map G corresponding to the reflection through the x-axis and apply the Change of Variables Formula.

Now, for (b)–(c), we write $\mathcal{D} = \mathcal{D}_+ \cup \mathcal{D}_-$ and note that $\mathcal{D}$ remains the same under the reflection through the x-axis.

(b) Assume that $f(x,-y) = -f(x,y)$ always hold. Then show that

$$\iint_{\mathcal{D}} f(x,y)\,\mathrm{d}x\mathrm{d}y = 0.$$

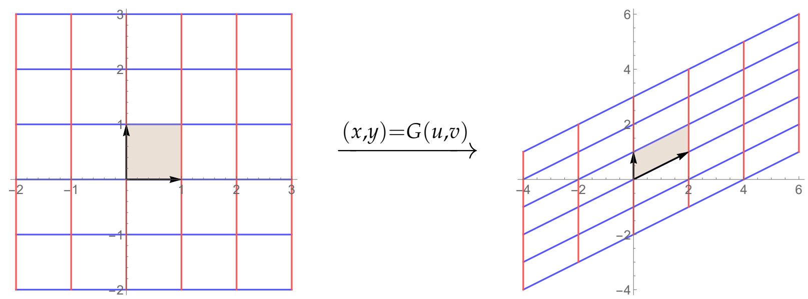
(c) In this case, assume that $f(x,-y) = f(x,y)$ always hold. Then show that

$$\iint_{\mathcal{D}} f(x,y)\,\mathrm{d}x\mathrm{d}y = 2\iint_{\mathcal{D}_+} f(x,y)\,\mathrm{d}x\mathrm{d}y.$$

Solution of some selected problems

Solution of Exercise 16.6.1, using the linearity.

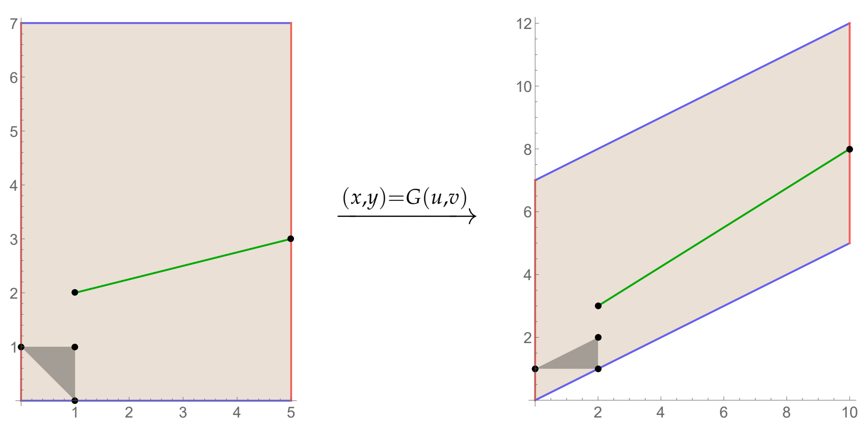
(a) Since G is a linear map, we know that G maps lines to lines. Since the u-axis is the line passing through $(0,0)$ and $(1,0)$ in the uv-plane, its image under G is the line passing through $G(0,0) = (0,0)$ and $G(1,0) = (2,1)$ in the xy-plane, or equivalently, $y = \frac{1}{2}x$. Likewise, the v-axis passes through $(0,0)$ and $(0,1)$, and so, its image under G is the line passing through $G(0,0) = (0,0)$ and $G(0,1) = (0,1)$. This is the y-axis in the xy-plane.



(b) Since G maps parallel lines to parallel lines, the image of $\mathcal{R}$ must be a parallelogram. So it suffices to determine the corners of $G(\mathcal{R})$, which are $G(0,0) = (0,0)$, $G(5,0) = (10,5)$, $G(5,7) = (10,12)$, and $G(0,7) = (0,7)$.

(c) As before, the image of the line segment in the uv-plane joining $(1,2)$ and $(5,3)$ is the line segment in the xy-plane joining $G(1,2) = (2,3)$ and $G(5,3) = (10,8)$.

(d) The image of the triangle in the uv-plane with vertices $(0,1)$, $(1,0)$ and $(1,1)$ is the triangle in the xy-plane with vertices $G(0,1) = (0,1)$, $G(1,0) = (2,1)$ and $G(1,1) = (2,2)$.



Solution of Exercise 16.6.1, using the general approach.

(a) The points on the u-axis can be parametrized by $(u,0)$ where u can take any values in $\mathbb{R}$. The image of these points are of the form $(x,y) = G(u,0) = (2u,u)$. This parametrizes the line $y = \frac{1}{2}x$ passing through the origin with the slope $\frac{1}{2}$.

Likewise, the points on the v-axis may be parametrized by $(0, v)$ for $v \in \mathbb{R}$ and they are mapped under G to $(x, y) = G(0, v) = (0, v)$. This parametrizes the line $x = 0$, which is the y-axis.

(b) The inverse map G^{-1} is given by $(u, v) = G^{-1}(x, y) = \left(\frac{x}{2}, y - \frac{x}{2}\right)$. Since $\mathcal{R}$ is given by the set of equations $0 \le u \le 5$ and $0 \le v \le 7$, plugging this inverse map gives $0 \le \frac{x}{2} \le 5$ and $0 \le y - \frac{x}{2} \le 7$, or equivalently,

$$0 \le x \le 10 \qquad \text{and} \qquad \frac{x}{2} \le y \le 7 + \frac{x}{2}.$$

This set of inequalities describe the image $G(\mathcal{R})$.

Alternatively, note that $\mathcal{R}$ is surrounded by the four lines $u = 0$, $u = 5$, $v = 0$, and $v = 7$. Arguing as before, these lines transform to lines $x = 0$, $x = 10$, $y = \frac{1}{2}x$, and $y = \frac{1}{2}x + 7$. These lines enclose the image $G(\mathcal{R})$.

Finally, in both answers, the image $G(\mathcal{R})$ corresponds to the parallelogram with vertices $(0, 0)$, $(10, 5)$, $(10, 12)$, and $(0, 7)$.

(c) The line segment is parametrized by $(u, v) = (1 + 4t, 2 + t)$ for $0 \le t \le 1$. Its image under G is then $(x, y) = G(1 + 4t, 2 + t) = (2 + 8t, 3 + 5t)$, which parametrizes the line segment joining $(2, 3)$ and $(10, 8)$ in the xy-plane.

(d) *(Skipped)*

$\square$

Solution of Exercise 16.6.16. We have

$$\text{Jac}(G) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{v}{u} & \ln u \\ \frac{2u}{v} & -\frac{u^2}{v^2} \end{vmatrix} = \left(\frac{v}{u}\right)\left(-\frac{u^2}{v^2}\right) - (\ln u)\left(\frac{2u}{v}\right) = -\frac{u(1 + \ln u)}{v}.$$

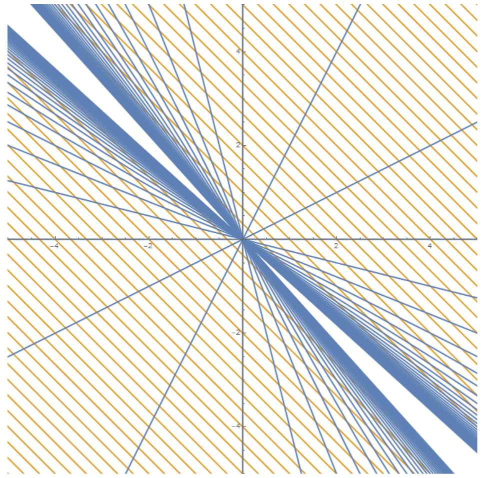
Now plug $(u, v) = (1, 2)$ to this. $\square$

Solution of Exercise 16.6.22.

(a) The horizontal line $v = c$ can be parametrized as (u, c) for $u \in \mathbb{R}$. Its image is then $(x, y) = G(u, c) = (u - cu, cu)$. So

- If $c \neq 1$, then $y = cu = c \cdot \frac{x}{1-c} = \frac{c}{1-c}x$.
- If $c = 1$, then $(x, y) = (0, u)$. Since u is arbitrary, this represent any point on the y-axis, hence the image of $v = 1$ is the y-axis.

(b) The vertical line $u = c$ can be parametrized as (c, v) for $v \in \mathbb{R}$, and so, its image is parametrized by $(x, y) = G(c, v) = (c - cv, cv)$. This corresponds to the line $x + y = c$.



The images of horizontal lines (blue) and vertical lines (orange).

(c) The Jacobian is computed as

$$\operatorname{Jac}(G) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = (1-v)u - (-u)v = u.$$

Remark. *Before moving to the remaining parts (d)–(e), we remark that G is one-to-one with the inverse map*

$$(u,v) = G^{-1}(x,y) = \left(x+y, \frac{y}{x+y} \right).$$

So it totally makes sense to discuss the image of $\mathcal{D}$ under G^{-1} as well as the Change of Variables Formula.

(d) The region $\mathcal{D}$ is bounded by four lines $y = 0$, the y-axis, $y + x = a$, and $y + x = b$. Under the inverse map G^{-1}, they correspond to the lines $v = 0$, $v = 1$, $u = a$, and $u = b$. So the image of $\mathcal{D}$ under the inverse map G^{-1} is the rectangle $\mathcal{D}_0 = [a,b] \in [0,1]$, and the Change of Variables Formula tells:

$$\operatorname{area}(\mathcal{D}) = \iint_{\mathcal{D}} 1 \, dx dy = \iint_{\mathcal{D}_0} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv = \iint_{\mathcal{D}_0} u \, du dv.$$

The double integral in the right-hand side can be computed by Fubini's theorem:

$$\iint_{\mathcal{D}_0} u \, du dv = \int_a^b \int_0^1 u \, dv du = \int_a^b u \, du = \frac{b^2 - a^2}{2}.$$

(e) We already know how $\mathcal{D}$ converts back to another domain $\mathcal{D}_0$ in the uv-plane. Also, if we write $f(x,y) = xy$, then $f(x,y) = f(G(u,v)) = f(u - uv, uv)$. So

$$\begin{aligned}\iint_{\mathcal{D}} f(x,y)\,\mathrm{d}x\mathrm{d}y &= \iint_{\mathcal{D}_0} f(G(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\mathrm{d}v \\ &= \int_0^1 \int_a^b u^3 v(1-v)\,\mathrm{d}u\mathrm{d}v \\ &= \frac{b^4 - a^4}{24}.\end{aligned}$$

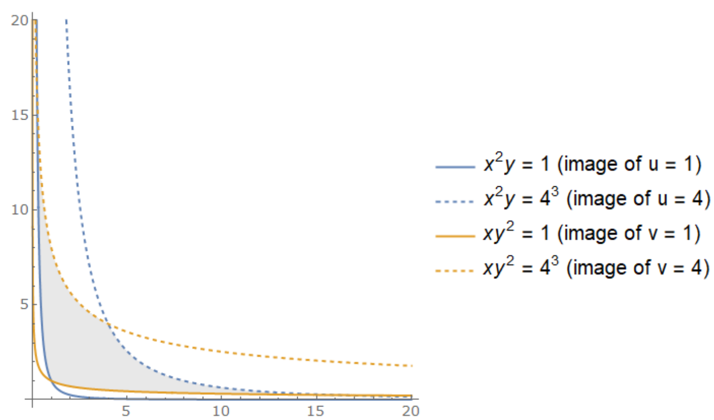
□

Solution of Exercise 16.6.30.

(a) The Jacobain of G is

$$\mathrm{Jac}(G) = \begin{vmatrix} 2u/v & -u^2/v^2 \\ -v^2/u^2 & 2v/u \end{vmatrix} = \left(\frac{2u}{v}\right)\left(\frac{2v}{u}\right) - \left(-\frac{u^2}{v^2}\right)\left(-\frac{v^2}{u^2}\right) = 3.$$

(b) The horizontal line $v = c$ in the uv-plane may be parametrized by (u, c), which is then mapped to $(x,y) = G(u,c) = (u^2/c, c^2/u)$ under G. Eliminating the parametrizing variable, this leads to the curve $xy^2 = c^3$ in the xy-plane. A similar argument shows that the vertical line $u = c$ is mapped to the curve $x^2y = c^3$. Since $\mathcal{R}$ is the region in the uv-plane surrounded by lines $u = 1$, $u = 4$, $v = 1$, and $v = 4$, the image of $\mathcal{R}$ under G is the region surrounded by the curves $x^2y = 1$, $x^2y = 4^3$, $xy^2 = 1$, and $xy^2 = 4^3$:



Sketch of the region $\mathcal{D}$.

(c) For the area of $\mathcal{D}$, the Change of Variables Formula together with $\mathrm{Jac}(G) = 3$ yields

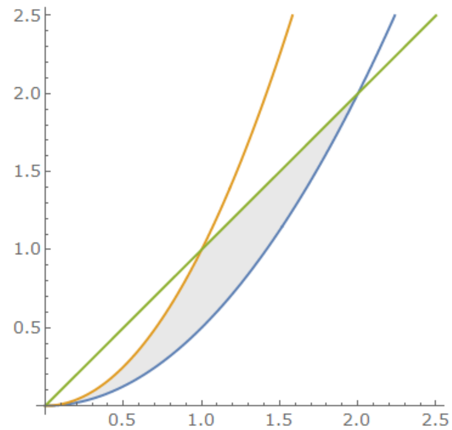
$$\begin{aligned}\mathrm{area}(\mathcal{D}) = \iint_{\mathcal{D}} 1\,\mathrm{d}x\mathrm{d}y &= \iint_{\mathcal{R}} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\mathrm{d}v \\ &= \iint_{\mathcal{R}} 3\,\mathrm{d}u\mathrm{d}v = 3\,\mathrm{area}(\mathcal{R}) = 27.\end{aligned}$$

Next, for the integral, the Change of Variables Formula shows that

$$\begin{aligned}\iint_{\mathcal{D}} (x+y)\, \mathrm{d}x\mathrm{d}y &= \iint_{\mathcal{R}} \left(\frac{u^2}{v} + \frac{v^2}{u}\right) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\mathrm{d}v \\ &= \int_1^4 \int_1^4 3\left(\frac{u^2}{v} + \frac{v^2}{u}\right) \mathrm{d}u\mathrm{d}v \\ &= 126 \ln 4.\end{aligned}$$

□

Solution of Exercise 16.6.37. The domain is as in the following figure:



Sketch of the region $\mathcal{D}$.

We will apply the Change of Variables Formula to compute the given integral.

Step 1. In order to do so, we first identify an appropriate map $(x, y) = G(u, v)$ for which the formula is applicable and then bring $\mathcal{D}$ back to the uv-plane. Let G be the function fron the first quadrant of the uv-plane to the xy-plane defined by

$$(x, y) = G(u, v) = (uv, u^2).$$

Here, the choice of domain ensures that G is one-to-one. Also, its inverse image is given by

$$(u, v) = G^{-1}(x, y) = \left(\sqrt{y}, \frac{x}{\sqrt{y}}\right).$$

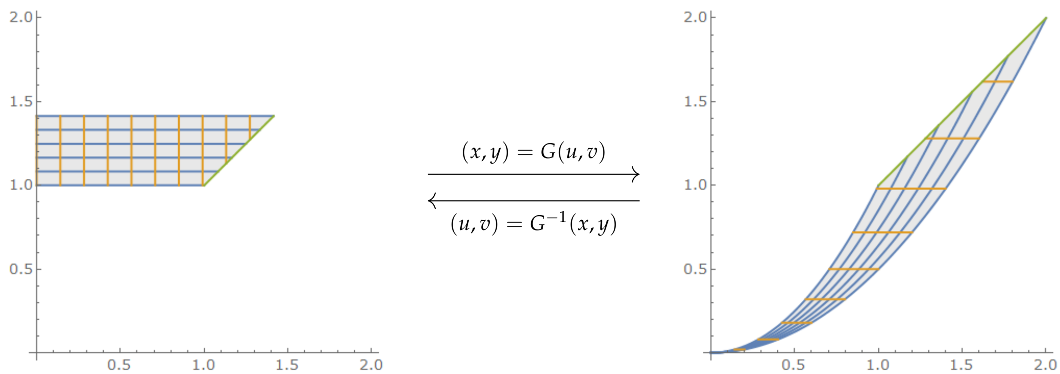
Here, the square root makes sense because the range of G is also the first quadrante of the xy-plane, and in particular, $x > 0$ and $y > 0$. Then the bounding curves in the xy-plane become

$$\begin{aligned} y = \tfrac{1}{2}x^2 &\quad \xrightarrow{(u,v)=G^{-1}(x,y)} \quad v = \sqrt{2}, \\ y = x^2 &\quad \xrightarrow{(u,v)=G^{-1}(x,y)} \quad v = 1, \\ y = x &\quad \xrightarrow{(u,v)=G^{-1}(x,y)} \quad v = u. \end{aligned}$$

From this, we deduce that the image of $\mathcal{D}_0$ under the inverse map G^{-1} is the region surrounded by $v = 1$, $v = \sqrt{2}$, and $v = u$, or equivalently, the region defined by

$$1 \le v \le \sqrt{2} \qquad \text{and} \qquad u \le v.$$

The following figure demonstrates this correspondence.



The region $\mathcal{D}_0$ in the uv-plane. The region $\mathcal{D}$ in the xy-plane.

Step 2. Next, we compute the Jacobian of G.

$$\operatorname{Jac}(G) = \begin{vmatrix} v & u \\ 2u & 0 \end{vmatrix} = -2u^2.$$

(*Remark*: The negative sign here correctly reflects the fact that G not only distorts the region $\mathcal{D}_0$ but also flips it.)

Step 3. Now that we have determined the region $\mathcal{D}_0$ and the Jacobian $\operatorname{Jac}(G)$, we apply the Change of Variables Formula to compute the integral.

$$\iint_{\mathcal{D}} \frac{1}{y}\,\mathrm{d}x\mathrm{d}y = \iint_{\mathcal{D}_0} \frac{1}{u^2} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \mathrm{d}u\mathrm{d}v = \iint_{\mathcal{D}_0} 2\,\mathrm{d}u\mathrm{d}v = \int_1^{\sqrt{2}} \int_0^v 2\,\mathrm{d}u\mathrm{d}v = 1.$$

Alternatively, this can be computed by noting that the double integral over $\mathcal{D}_0$ is twice the area of the trapezoid $\mathcal{D}_0$. □