Homework 3

- Homework 3 is due January 24 in class.
- Exercises are taken from J. Rogawski, C. Adams, R. Franzosa *Calculus, Multivariable*, 4th Ed., W. H. Freeman & Company.
- The starred problems will not be graded.

16.4. Integration in Polar, Cylindrical, and Spherical Coordinates

Exercises

In Exercises 3 and 5, sketch the region $\mathcal{D}$ indicated and integrate $f(x,y)$ over $\mathcal{D}$ using polar coordinates.

3. $f(x,y) = xy; \quad x \geq 0,\ y \geq 0,\ x^2+y^2 \leq 4$

5. $f(x,y) = y(x^2+y^2)^{-1}; \quad y \geq \frac{1}{2},\ x^2+y^2 \leq 1$

In Exercises 9, 12, and 13, sketch the region of integration and evaluate by changing to polar coordinates.

9. $\displaystyle\int_0^{1/2} \int_{\sqrt{3}x}^{\sqrt{1-x^2}} x \, dy dx$

12.* $\displaystyle\int_0^{2} \int_{x}^{\sqrt{3}x} y \, dy dx$

13. S$\displaystyle\int_{-1}^{2} \int_{0}^{\sqrt{4-x^2}} (x^2+y^2) \, dy dx$

In Exercises 18–20, calculate the integral over the given region by changing to polar coordinates.

18. $f(x,y) = (x^2+y^2)^{-3/2}; \quad x^2+y^2 \leq 1,\ x+y \geq 1$

19.* $f(x,y) = x - y; \quad x^2+y^2 \leq 1,\ x+y \geq 1$

20. $f(x,y) = y; \quad x^2+y^2 \leq 1,\ (x-1)^2+y^2 \leq 1$

21. Find the volume of the wedge-shaped region (Figure 18) contained in the cylinder $x^2+y^2=9$ bounded above by the plane $z=x$ and below by the xy-plane.

Hint: Use Fubini's Theorem to write the volume as $\displaystyle\iint_{\mathcal{D}} \int_0^x 1 \, dz dA$ for an appropriate domain $\mathcal{D}$ in the xy-plane.

22. Let $\mathcal{W}$ be the region above the sphere $x^2+y^2+z^2=6$ and below the paraboloid $z = 4 - x^2 - y^2$.

(a) Show that the projection of $\mathcal{W}$ on the xy-plane is the disk $x^2+y^2 \leq 2$ (Figure 19).

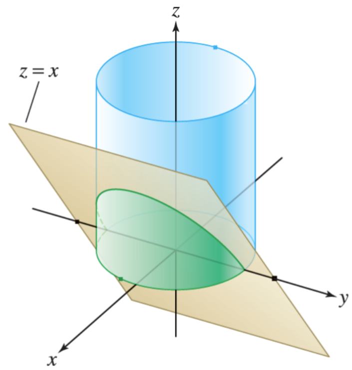

Figure 18

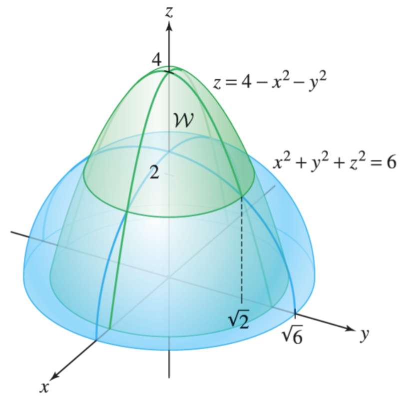

Figure 19

(b) Compute the volume of using polar coordinates.

23.[⋆] Evaluate $\iint_{\mathcal{D}} \sqrt{x^2+y^2}\,\mathrm{d}A$ where $\mathcal{D}$ is the domain in Figure 20.
Hint: Find the equation of the inner circle in polar coordinates and treat the right and left parts of the region separately.

24. Evaluate $\iint_{\mathcal{D}} x\sqrt{x^2+y^2}\,\mathrm{d}A$ where $\mathcal{D}$ is the shaded region enclosed by the lemniscate curve $r^2 = \sin 2\theta$ in Figure 21.

In Exercises 29 and 32, use cylindrical coordinates to calculate $\iiint_{\mathcal{W}} f(x,y,z)\,\mathrm{d}V$ *for the given function and region.*

29. $f(x,y,z) = x; \quad x^2+y^2 \le 16,\ x \ge 0,\ y \ge 0,\ -3 \le z \le 3$

32. $f(x,y,z) = z; \quad 0 \le z \le x^2+y^2 \le 9$

35. Express the triple integral $\int_{-1}^{1}\int_{y=0}^{y=\sqrt{1-x^2}}\int_{z=0}^{z=x^2+y^2} f(x,y,z)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x$ in cylindrical coordinates.

37. Find the equation of the right-circular cone in Figure 22 in cylindrical coordinates and compute its volume.

In Exercises 4, 48, and 50, use spherical coordinates to calculate the triple integral of $f(x,y,z)$ *over the given region.*

45. $f(x,y,z) = y; \quad x^2+y^2+z^2 \le 1,\ x,y,z \le 0$

48. $f(x,y,z) = 1; \quad x^2+y^2+z^2 \le 4z,\ z \ge \sqrt{x^2+y^2}$

50. $f(x,y,z) = \rho; \quad x^2+y^2+z^2 \le 4,\ z \le 1,\ x \ge 0$

52. Find the volume of the region lying above the cone $\phi = \phi_0$ and below the sphere $\rho = R$.

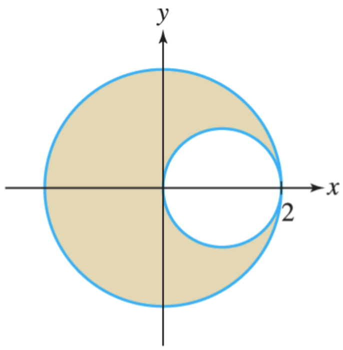

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Figure 20

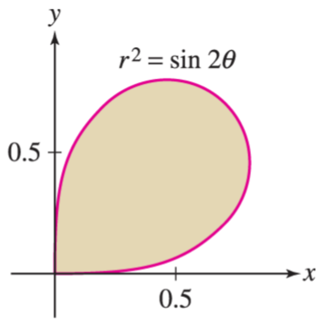

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Figure 21

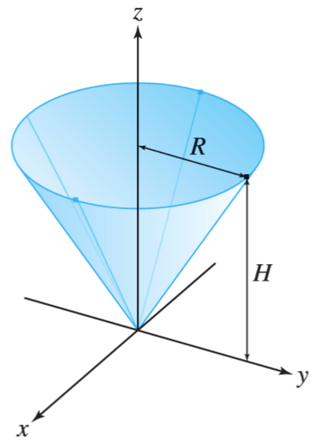

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Figure 22

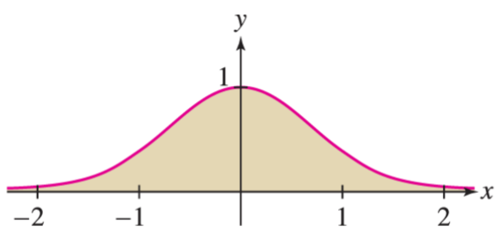

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Figure 24. The bell-shaped curve $y = e^{-x^2}$.

54. Calculate the volume of the cone in Figure 22, using spherical coordinates.

56. Let $\mathcal{W}$ be the region within the cylinder $x^2 + y^2 = 2$ between $z = 0$ and the cone $z = \sqrt{x^2 + y^2}$. Calculate the integral of $f(x, y, z) = x^2 + y^2$ over $\mathcal{W}$, using both spherical and cylindrical coordinates.

57.* **Bell-Shaped Curve** One of the key results in calculus is the computation of the area under the bell-shaped curve (Figure 24):

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$$

This integral appears throughout engineering, physics, and statistics, and although e^{-x^2} does not have an elementary antiderivative, we can compute using multiple integration.

(a) Show that $I^2 = J$, where J is the improper double integral

$$J = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2}\, dx dy$$

Hint: Use Fubini's Theorem and $e^{-x^2-y^2} = e^{-x^2}e^{-y^2}$.

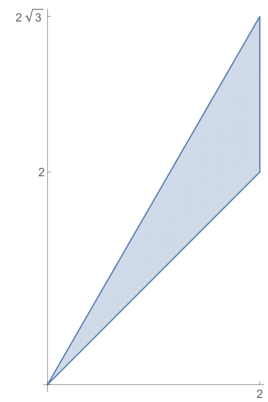
(b) b. Evaluate in polar coordinates.

(c) Prove that $I = \sqrt{\pi}$.

Solutions of some selected problems

Solution of Exercise 12. The region of integration $\mathcal{D}$ is described by

$$0 \le x \le 2 \qquad \text{and} \qquad x \le y \le \sqrt{3}x.$$



In other words, $\mathcal{D}$ is bounded by $y = x$, $y = \sqrt{3}x$, and $x = 2$. Since these lines are converted to polar equations $\theta = \frac{\pi}{4}$, $\theta = \frac{\pi}{3}$, and $r = 2\sec\theta$, respectively, we may regard $\mathcal{D}$ as a radially simple region described by

$$\frac{\pi}{4} \le \theta \le \frac{\pi}{3} \qquad \text{and} \qquad 0 \le r \le 2\sec\theta.$$

Moreover, the integrand $f(x,y) = y$ is converted, in polar coordinates, to $f(r\sin\theta, r\cos\theta) = r\sin\theta$. Therefore So the iterated integral in question is computed as:

$$\begin{aligned} \int_0^2 \int_x^{\sqrt{3}x} y\,\mathrm{d}y\mathrm{d}x &= \iint_{\mathcal{D}} y\,\mathrm{d}A = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \int_0^{2\sec\theta} r\sin\theta\, r\mathrm{d}r\mathrm{d}\theta \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\int_0^{2\sec\theta} r^2\,\mathrm{d}r \right) \sin\theta\,\mathrm{d}\theta. \end{aligned} \tag{16.1}$$

The inner integral in r is easily computed as

$$\int_0^{2\sec\theta} r^2\,\mathrm{d}r = \left[\frac{r^3}{3} \right]_{r=0}^{2\sec\theta} = \frac{8}{3\cos^3\theta}.$$

Plugging this back to (16.1) and proceeding,

$$(16.1) = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{8}{3\cos^3\theta} \sin\theta\,\mathrm{d}\theta = \left[\frac{4}{3\cos^2\theta} \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{8}{3}$$

□

Solution of Exercise 19. Before proceeding to the solution, we remark that the integral is zero by the symmetry argument. Indeed, consider the reflection $(x,y) \mapsto (y,x)$ around the line $y = x$.

Let $\mathcal{D}$ denote the domain of integration. This is the region bounded by $x^2 + y^2 = 1$ and $x + y = 1$. Converting these to polar equations, we get $r = 1$ and $r = (\cos\theta + \sin\theta)^{-1}$. This allows to describe $\mathcal{D}$ as a radially simple region:

$$\mathcal{D}: \qquad 0 \le \theta \le \frac{\pi}{2} \qquad \text{and} \qquad \frac{1}{\cos\theta + \sin\theta} \le r \le 1.$$

Together with $f(r\cos\theta, r\sin\theta) = r(\cos\theta - \sin\theta)$, we get

$$\begin{aligned}\iint_{\mathcal{D}} (x-y)\,\mathrm{d}A &= \int_0^{\frac{\pi}{2}} \int_{\frac{1}{\cos\theta+\sin\theta}}^{1} r(\cos\theta-\sin\theta)\, r\mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{2}} \left(\int_{\frac{1}{\cos\theta+\sin\theta}}^{1} r^2\,\mathrm{d}r\right)(\cos\theta-\sin\theta)\,\mathrm{d}\theta\end{aligned} \tag{16.2}$$

The inner integral in r is:

$$\int_{\frac{1}{\cos\theta+\sin\theta}}^{1} r^2\,\mathrm{d}r = \left[\frac{r^3}{3}\right]_{r=\frac{1}{\cos\theta+\sin\theta}}^{1} = \frac{1}{3}\left(1 - \frac{1}{(\cos\theta+\sin\theta)^3}\right).$$

Plugging this back, we get

$$(16.2) = \int_0^{\frac{\pi}{2}} \frac{1}{3}\left(1 - \frac{1}{(\cos\theta+\sin\theta)^3}\right)(\cos\theta-\sin\theta)\,\mathrm{d}\theta. \tag{16.3}$$

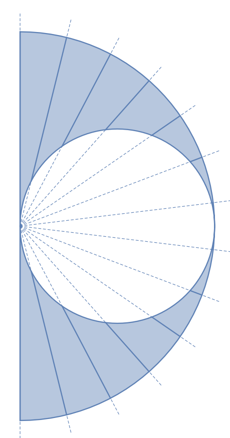
This integral can be easily computed by substituting $u = \cos\theta + \sin\theta$. Indeed, by noting that $\mathrm{d}u = (\cos\theta - \sin\theta)\,\mathrm{d}\theta$, the integral (16.3) becomes

$$\int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{1}{3}\left(1 - \frac{1}{(\cos\theta+\sin\theta)^3}\right)(\cos\theta-\sin\theta)\,\mathrm{d}\theta = \int_{u=1}^{u=1} \frac{1}{3}\left(1-\frac{1}{u^3}\right)\mathrm{d}u = 0.$$

□

Solution of Exercise 23. The region $\mathcal{D}$ is bounded between two circles

$$x^2+y^2 = 4 \qquad \text{and} \qquad (x-1)^2+y^2 = 1.$$



In polar coordinates, the larger circle is easily described as $r = 2$. The case of the smaller circle is a bit tricky. First, since it lies in the right half-plane, we have to restrict the range of θ describing this circle to $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. After than, applying the conversion formula $(x,y) = (r\cos\theta, r\sin\theta)$ to the equation of the smaller circle becomes:

$$(x-1)^2+y^2 = 1 \qquad \Leftrightarrow \qquad x^2+y^2 = 2x \qquad \Leftrightarrow \qquad r = 2\cos\theta.$$

This tells that the part of the region $\mathcal{D}$ corresponding to $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ is a radially simple region bounded between two curves $r = 2$ and $r = 2\cos\theta$.

Now write $\mathcal{D}_{\text{right}}$ ($\mathcal{D}_{\text{left}}$, respectively) for the part of $\mathcal{D}$ lying in the right half-plane (left half-plane, respectively). Then $\mathcal{D}_{\text{left}}$ is the polar rectangle

$$\mathcal{D}_{\text{left}} \ : \qquad \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \qquad \text{and} \qquad 0 \leq r \leq 2$$

and $\mathcal{D}_{\text{right}}$ is the radially simple region

$$\mathcal{D}_{\text{right}} \ : \qquad \frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2} \qquad \text{and} \qquad 2\cos\theta \leq r \leq 2.$$

Together with $\sqrt{x^2+y^2} = r$, it then follows that

$$\begin{aligned}
\iint_{\mathcal{D}} \sqrt{x^2+y^2}\,\mathrm{d}A &= \iint_{\mathcal{D}_{\text{left}}} \sqrt{x^2+y^2}\,\mathrm{d}A + \iint_{\mathcal{D}_{\text{right}}} \sqrt{x^2+y^2}\,\mathrm{d}A \\
&= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^2 r^2\,\mathrm{d}r\mathrm{d}\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{2\cos\theta}^2 r^2\,\mathrm{d}r\mathrm{d}\theta \\
&= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{8}{3}\,\mathrm{d}\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8}{3}(1-\cos^3\theta)\,\mathrm{d}\theta
\end{aligned}$$

The first integral in the last line is easily computed as $\frac{8\pi}{3}$. For the second one, we may utilize the identity $\cos^2\theta + \sin^2\theta = 1$ and the substitution $u = \sin\theta$ to compute:

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8}{3}(1-\cos^3\theta)\,\mathrm{d}\theta &= \frac{8\pi}{3} - \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\sin^2\theta)\cos\theta\,\mathrm{d}\theta \\
&= \frac{8\pi}{3} - \frac{8}{3}\int_{-1}^{1} (1-u^2)\,\mathrm{d}u \\
&= \frac{8\pi}{3} - \frac{32}{9}.
\end{aligned}$$

Therefore the answer is $\dfrac{16\pi}{3} - \dfrac{32}{9}$. $\square$