Section 5.1 Functions of One Random Variables.

<u>GOAL</u> Knowing the dist. of X, how to determine the dist. of $Y = u(X)$?

1 Case : X is discrete

(1) In this case, $Y = u(X)$ is always discrete, and

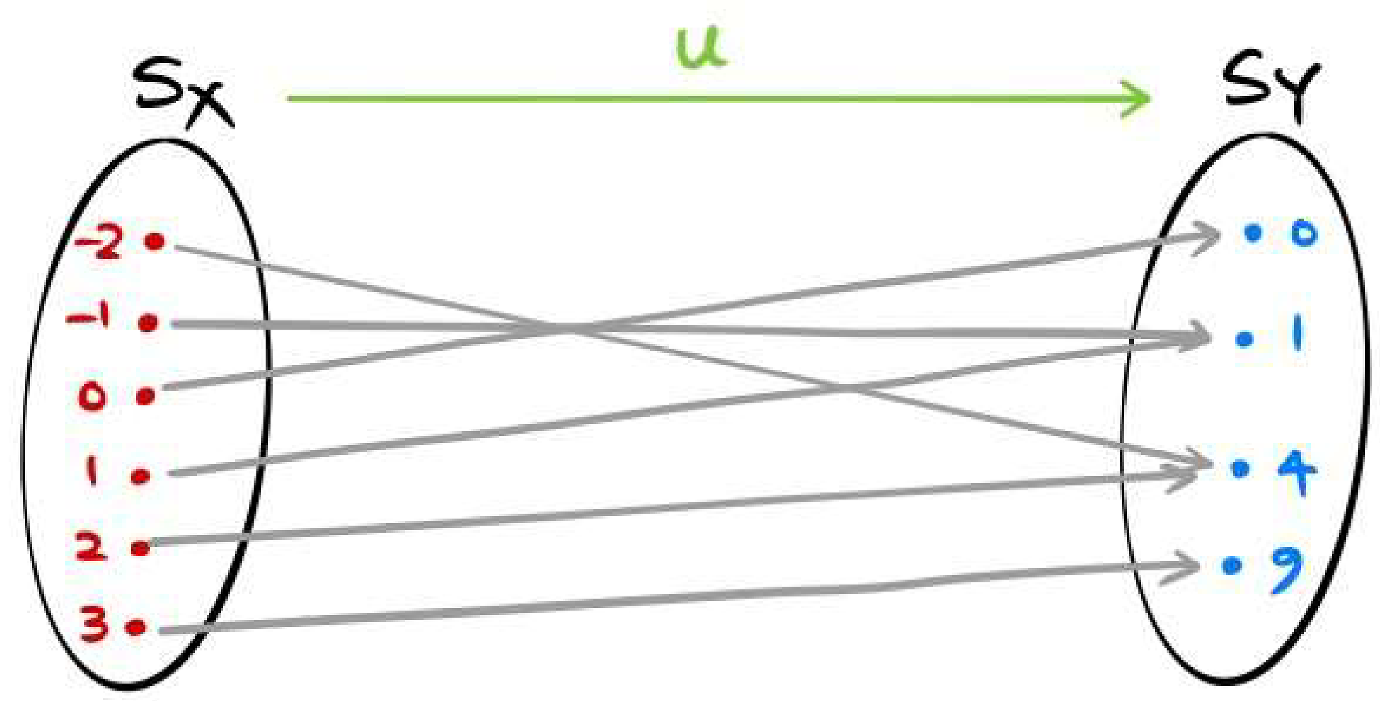
$$p_Y(y) = P(Y=y) = P(u(X)=y) = \sum_{x:\,u(x)=y} P(X=x) = \sum_{x:\,u(x)=y} p_X(x).$$

(2) Special case : If $u(x)$ is one-to-one on S_X, then

$$p_Y(y) = p_X(u^{-1}(y)).$$

Ex Let $\begin{cases} X \text{ has a PMF } p_X(x) = \frac{1}{6} & x = -2, 1, 0, 1, 2, 3 \\ u(x) = x^2. \end{cases}$

Then



$$p_Y(y) = \sum_{x:\,x^2=y} p_X(x) = \begin{cases} \frac{1}{6}, & \text{if } y = 0, 9 \\ \frac{2}{6}, & \text{if } y = 1, 4 \end{cases} \qquad \square$$

Ex Let X : discrete RV and $Y = -3X + 7$. Then

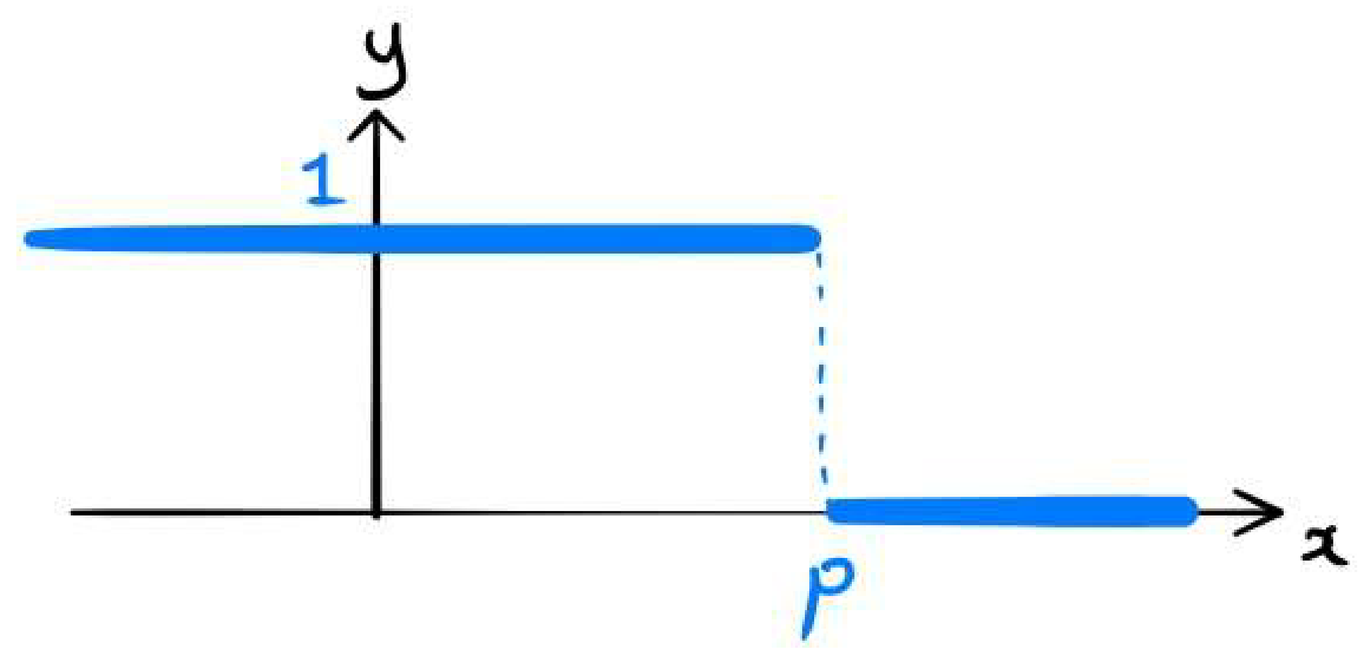
$$p_Y(y) = P(-3X+7=y) = P\left(X = \frac{7-y}{3}\right) = p_X\left(\frac{7-y}{3}\right). \qquad \square$$

$\boxed{2}$ Case : X is continuous.

- In this case, it is not guaranteed that $Y = u(X)$ is continuous.

Ex Let $p \in [0,1]$ and

$$u(x) = \begin{cases} 1, & x < p, \\ 0, & x \geq p. \end{cases}$$



graph of $y = u(x)$

If X is $\mathcal{U}(0,1)$, then $Y = u(X)$ takes only two values $0, 1$, and

$$p_Y(0) = P(Y=0) = P(X \geq p) = 1-p,$$
$$p_Y(1) = P(Y=1) = P(X < p) = p.$$

So Y has a Bernoulli dist. w/ parameter p. $\square$

- If both X and $Y = u(X)$ are continuous RVs, we may relate $f_X(x)$ and $f_Y(y)$ by "CDF Technique":

(1) **CDF Technique**: Write

$$F_Y(y) = P(Y \leq y) = P(u(X) \leq y)$$
$$= P(\text{solving } u(X) \leq y \text{ in terms of } X)$$

and differentiate both sides w.r.t. y.

(2) Monotone Case : **Change-of-Variables technique**

Suppose $\left\{ \begin{array}{ll} S_X & : \text{interval}, \\ u(x) & : \text{monotone} \end{array} \right\}$. Write $v = u^{-1}$: inverse fn. Then for $y \in S_Y$,

$$F_Y(y) = P(Y \le y) = P(u(X) \le y)$$

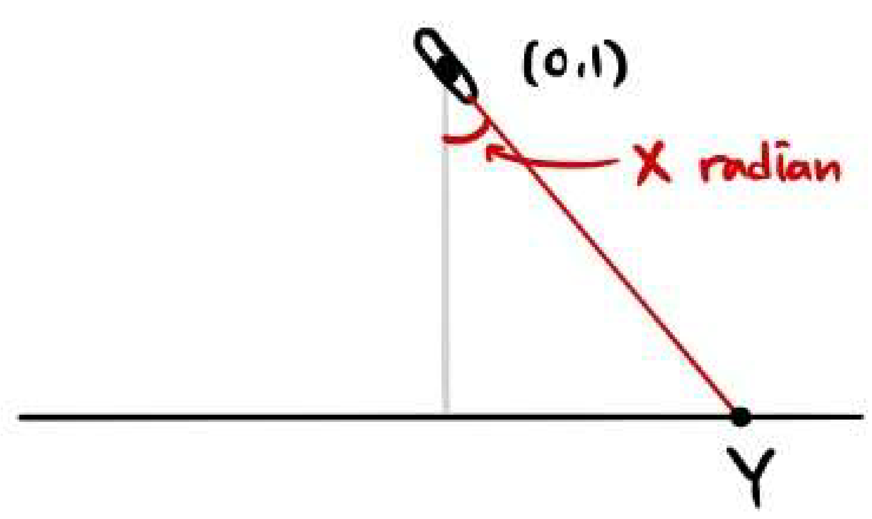
$$= \begin{cases} P(X \le v(y)) = F_X(v(y)), & \text{if } u \text{ is increasing,} \\ P(X \ge v(y)) = 1 - F_X(v(y)), & \text{if } u \text{ is decreasing.} \end{cases}$$

Differentiating w.r.t. y,

$$f_Y(y) = \begin{cases} f_X(v(y))\, v'(y), & u \text{ inc.} \\ -f_X(v(y))\, v'(y), & u \text{ dec.} \end{cases}$$

$$= |v'(y)|\, f_X(v(y)) = \frac{1}{|u'(x)|} f_X(x), \text{ where } u(x) = y.$$

Ex (Cauchy distribution) Let X be $\mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2})$ and consider:



Then $Y = u(X)$ with $u(x) = \tan(x)$. So $v(y) = u^{-1}(y) = \arctan(y)$, and

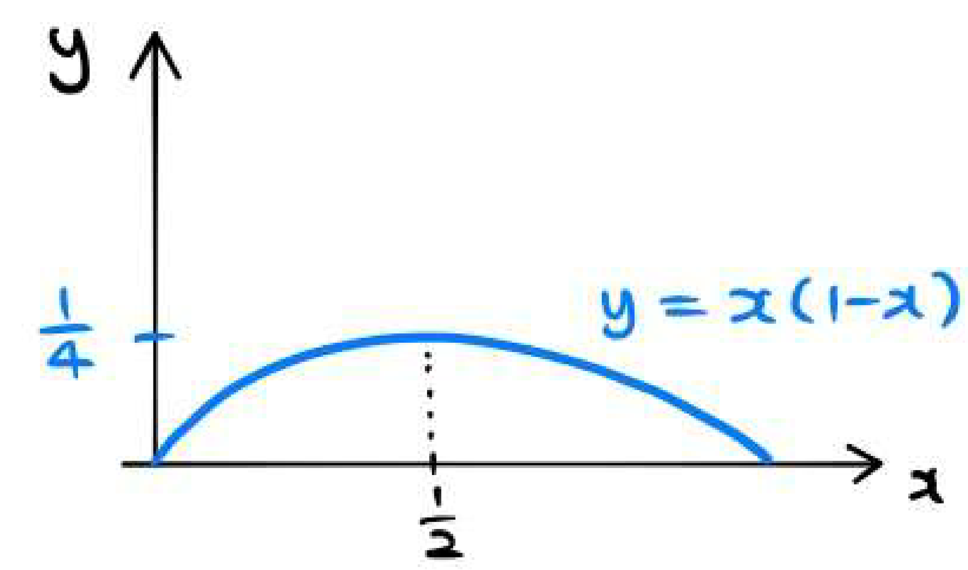
- ▷ $S_Y = (-\infty, \infty)$.
- ▷ $f_Y(y) = |v'(y)|\, f_X(v(y)) = \dfrac{1}{1+y^2} \cdot \dfrac{1}{\pi} = \dfrac{1}{\pi(1+y^2)}$. $\square$

Ex Let X has an exponential dist. w/ mean θ, and $Y = e^{-X/\theta}$. Then we may write $Y = u(X)$ with $u(x) = e^{-x/\theta}$ $\Rightarrow$ $v(y) = u^{-1}(y) = -\theta \ln y$.
Then

- ▷ $S_Y = (0, 1)$,
- ▷ For each $y \in S_Y$, $f_Y(y) = |v'(y)|\, f_X(v(y)) = \left|-\frac{\theta}{y}\right| \frac{1}{\theta} e^{-(-\theta \ln y)/\theta} = 1$.

So Y has the uniform dist. $\mathcal{U}(0,1)$. $\square$

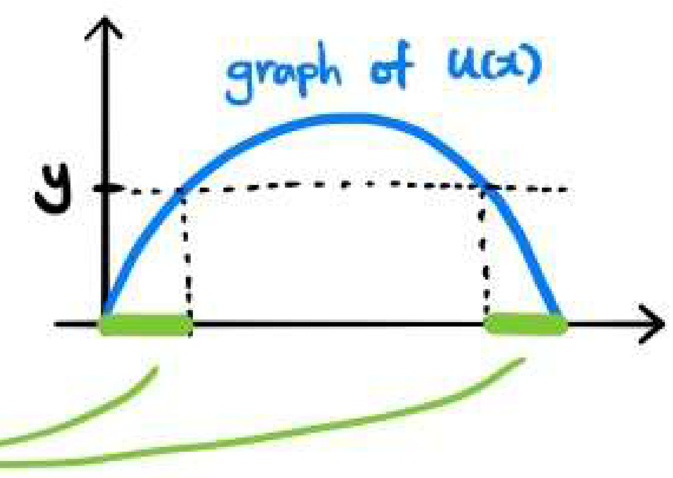
(3) Non-monotone Case :



Ex Let X be $\mathcal{U}(0,1)$ and $Y = X(1-X)$. Then

- ▷ $S_Y = [0, \frac{1}{4}]$.
- ▷ For each $y \in S_Y$,

$$F_Y(y) = P(Y \le y)$$
$$= P(X \in \{x : u(x) \le y\})$$



$$= P\left(X \le \frac{1-\sqrt{1-4y}}{2}\right) + P\left(X \ge \frac{1+\sqrt{1-4y}}{2}\right).$$
$$= F_X\left(\frac{1-\sqrt{1-4y}}{2}\right) + 1 - F_X\left(\frac{1+\sqrt{1-4y}}{2}\right).$$

Differentiating,

$$f_Y(y) = \left(\frac{1}{\sqrt{1-4y}}\right)\cdot(1) + 0 - \left(-\frac{1}{\sqrt{1-4y}}\right)\cdot(1) = \frac{2}{\sqrt{1-4y}}. \qquad \square$$

Remark) This computation generalizes to: If $Y = u(X)$ is continuous RV, then

$$f_Y(y) = \sum_{x : u(x) = y} \frac{1}{|u'(x)|} f_X(x).$$

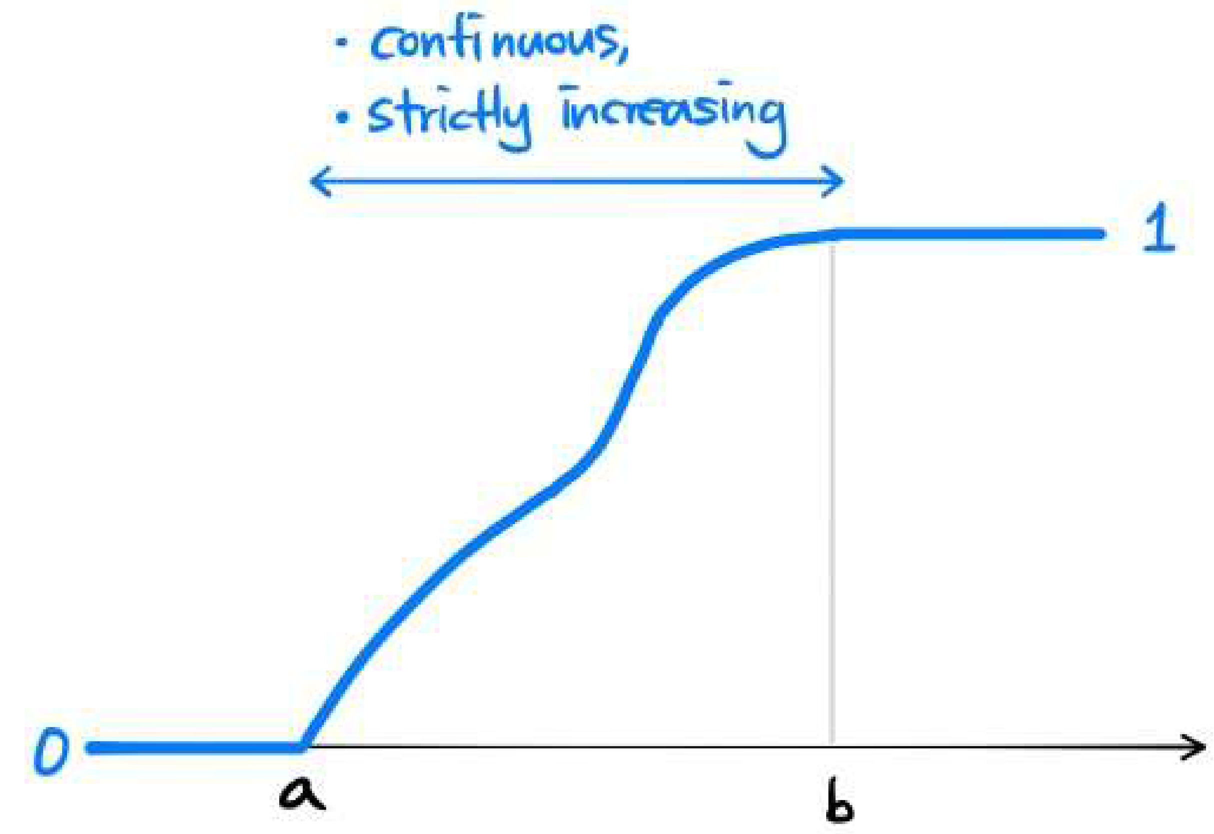
3 Simulating a distribution

Q Knowing the CDF of a distribution, how to simulate it?

THM Let:

- ▷ Y is $\mathcal{U}(0,1)$.
- ▷ $F(x)$: CDF of a continuous dist. with the support = interval (a, b).

Then $X = F^{-1}(Y)$ is continuous RV with CDF $F(x)$.



graph of F under the assumption

<u>Pf)</u> ▷ X may take only values in (a, b).

▷ For $a < x < b$,

$$\begin{aligned} F_X(x) &= P(X \le x) \\ &= P(F^{-1}(Y) \le x) \\ &= P(Y \le F(x)) \\ &= F_Y(F(x)) \\ &= F(x) \end{aligned}$$

$\because$ $\{F^{-1}(Y) \le x\}$ and $\{Y \le F(x)\}$ are logically equivalent

$\because$ $F_Y(y) = y$ if $0 < y < 1$.

□

<u>Remark)</u> With due modification, this theorem still holds for ANY CDF.

<u>Ex</u> (Simulating Exp. dist.) Let $\theta > 0$. Then

$$F(x) = \begin{cases} 0, & x \le 0, \\ 1 - e^{-x/\theta}, & 0 < x \end{cases}$$

is the CDF of Exp. dist. with mean θ. So, if Y is $\mathcal{U}(0,1)$,

$$X = -\theta \ln(1 - Y)$$

has CDF $F(x)$, i.e., X has Exp. dist with mean θ. □

- Tracing the proof backward, we also obtain:

<u>THM</u> Let X have a continuous dist. with the support = interval (a, b).
<u>Then</u> $Y = F_X(X)$ is $\mathcal{U}(0,1)$.