Note 20

Section 4.4 Bivariate Distributions of the Continuous Type

Last Time   We studied about joint PDFs.

DEF Let X and Y have the joint PDF $f(x,y)$.

- ▷ The conditional PDF of Y given $X=x$ is defined by
$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}, \qquad \text{provided } f_X(x) > 0.$$

- ▷ The conditional expectation of $u(Y)$ given $X=x$ is defined by
$$E[u(Y) \mid X=x] = \int_{-\infty}^{\infty} u(y)\, f_{Y|X}(y|x)\, dy.$$

- ▷ $E(Y|X)$ and $Var(Y|X)$ are defined as in the discrete case.

Rmk) Both the law of total exp / var hold in this case.

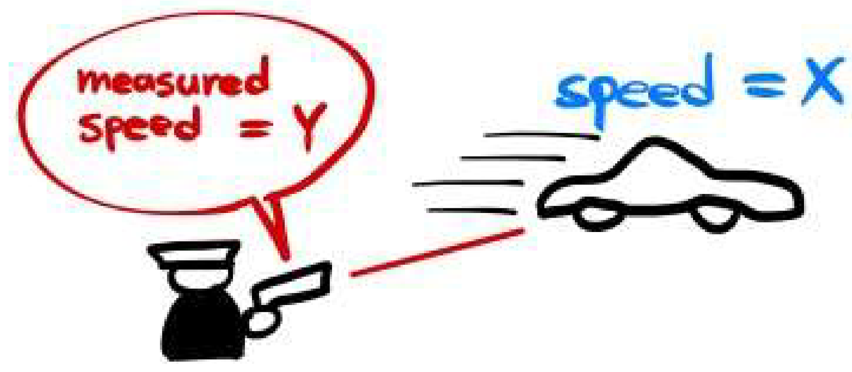
Ex Let X and Y be as in the previous ex:
$$f(x,y) = 2e^{-x-y}, \qquad 0 < y < x < +\infty$$

Then we can check that
$$f_X(x) = 2e^{-x}(1-e^{-x}), \qquad 0 < x < +\infty,$$
and so,
$$f_{Y|X}(y|x) = \frac{2e^{-x-y}}{2e^{-x}(1-e^{-x})} = \frac{e^{-y}}{1-e^{-x}}, \qquad 0 < y < x.$$

Then for $0 < x < +\infty$,
$$E(Y|X=x) = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|x)\, dy = \int_0^x \frac{ye^{-y}}{1-e^{-x}}\, dx$$
$$= \frac{1}{1-e^{-x}}\Big[-(y+1)e^{-y}\Big]_{y=0}^{y=x} = \frac{1}{1-e^{-x}}\left(1-(x+1)e^{-x}\right)$$
$$= 1 - \frac{x}{e^x - 1}.$$

∎

Ex A police officer is measuring the speed of cars on a highway.



- ▷ The actual speed X of a car is $\mathcal{U}(80, 120)$,
- ▷ Due to the inaccuracy of the speed gun, the speed Y measured, given $X = x$, is $\mathcal{N}(x, \frac{1}{100}x)$.

Find the mean / var of Y.

Sol) By the law of total E:

$$E(Y) = E[E(Y|X)] = E[X] = 100.$$

Likewise, by the law of total Var:

$$\begin{aligned} Var(Y) &= E(Var(Y|X)) + Var(E(Y|X)) \\ &= E\left(\frac{1}{100}X\right) + Var(X) \\ &= \frac{1}{100} \cdot 100 + \frac{1}{12} \cdot (120-80)^2 \\ &= 1 + \frac{400}{3}. \end{aligned}$$

$\square$

Section 4.5 Bivariate Normal Distribution

<u>GOAL</u> Get acquainted with 2D-analog of the normal distribution.

<u>Disclaimer</u> Since it is hard to appreciate the meaning & usefulness of this analog at this point, we will only briefly mention the definition and move on to the next topic.

DEF X and Y are said to have a bivariate normal distribution if they have the joint PDF of the form

$$f_{X,Y}(x,y) = \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1-\rho^2}} \exp\left(-\frac{1}{2} q(x,y)\right),$$

where

$$q(x,y) = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]$$

Remark) If you are familiar with matrices, you may write:

$$f_{X,Y}(\mathbf{x}) = \frac{1}{2\pi\sqrt{\det(\Sigma)}} \exp\left(-\frac{1}{2}\mathbf{x}^T \Sigma^{-1} \mathbf{x}\right), \qquad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

where

$$\Sigma = \begin{pmatrix} \mathrm{Cov}(X,X) & \mathrm{Cov}(X,Y) \\ \mathrm{Cov}(Y,X) & \mathrm{Cov}(Y,Y) \end{pmatrix}$$

is the covariance matrix.

Properties Let X and Y have the above bivariate normal PDF. Then

(1) Both X and Y are normal. More precisely:

$$X \text{ is } \mathcal{N}(\mu_X, \sigma_X^2) \quad \text{and} \quad Y \text{ is } \mathcal{N}(\mu_Y, \sigma_Y^2).$$

(2) Conditional distributions are normal:

Given $X = x$, Y is $\mathcal{N}\left(\mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X),\ \sigma_Y^2(1-\rho^2)\right)$

Given $Y = y$, X is $\mathcal{N}\left(\mu_X + \rho \frac{\sigma_X}{\sigma_Y}(y - \mu_Y),\ \sigma_X^2(1-\rho^2)\right)$.

(3) Corr. coef. of X and $Y = \rho$.

Ex Let X and Y are bivariate normal with $\mu_X = 70$, $\sigma_X^2 = 10^2$, $\mu_Y = 80$, $\sigma_Y^2 = 13^2$, $\rho = \frac{5}{13}$. Then

(1) The conditional dist. of Y, given $X = 76$, is

$$\mathcal{N}\left(80 + \left(\tfrac{5}{13}\right)\cdot \tfrac{13}{10}(76 - 70),\ 13^2\left(1 - \left(\tfrac{5}{13}\right)^2\right)\right) = \mathcal{N}(83, 12^2),$$

In particular,

$$E(Y \mid X = 76) = 83 \quad \text{and} \quad Var(Y \mid X = 76) = 12^2.$$

This is $\mathcal{N}(0,1)$ given $X = 76$.

(2) $P(Y \leq 86 \mid X = 76) = P\left(\frac{Y - 83}{12} \leq \frac{86 - 83}{12} \,\middle|\, X = 76\right)$

$$= \Phi\left(\frac{86-83}{12}\right) = \Phi\left(\frac{1}{4}\right).$$

$\square$

Section 5.1 Functions of One Random Variables.

<u>GOAL</u> Knowing the dist. of X, how to determine the dist. of $Y = u(X)$?

1 Case : X is discrete

(1) In this case, $Y = u(X)$ is always discrete, and

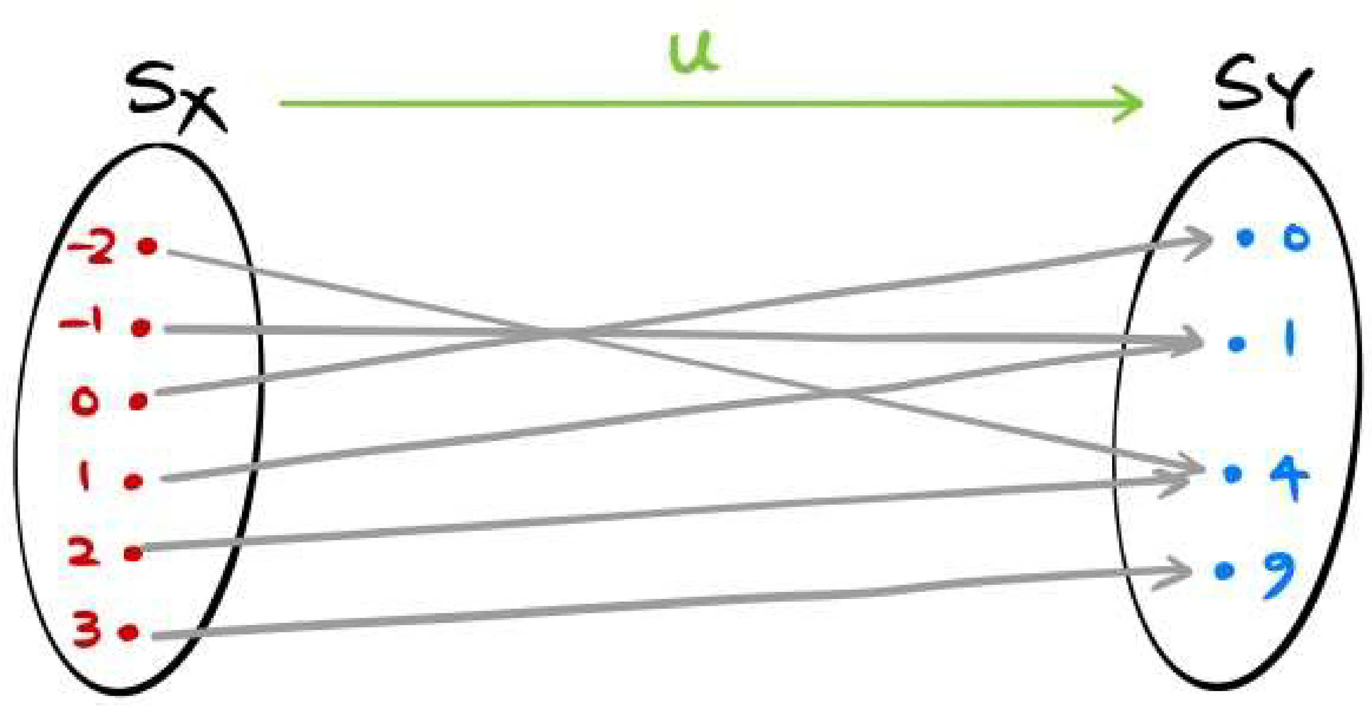
$$p_Y(y) = P(Y=y) = P(u(X)=y) = \sum_{x:\, u(x)=y} P(X=x) = \sum_{x:\, u(x)=y} p_X(x).$$

(2) Special case : If $u(x)$ is one-to-one on S_X, then

$$p_Y(y) = p_X(u^{-1}(y)).$$

Ex Let $\begin{cases} X \text{ has a PMF } p_X(x) = \frac{1}{6} & x = -2, 1, 0, 1, 2, 3 \\ u(x) = x^2. \end{cases}$

Then



$$p_Y(y) = \sum_{x:\, x^2=y} p_X(x) = \begin{cases} \frac{1}{6}, & \text{if } y = 0, 9 \\ \frac{2}{6}, & \text{if } y = 1, 4 \end{cases}$$ $\square$

Ex Let X : discrete RV and $Y = -3X + 7$. Then

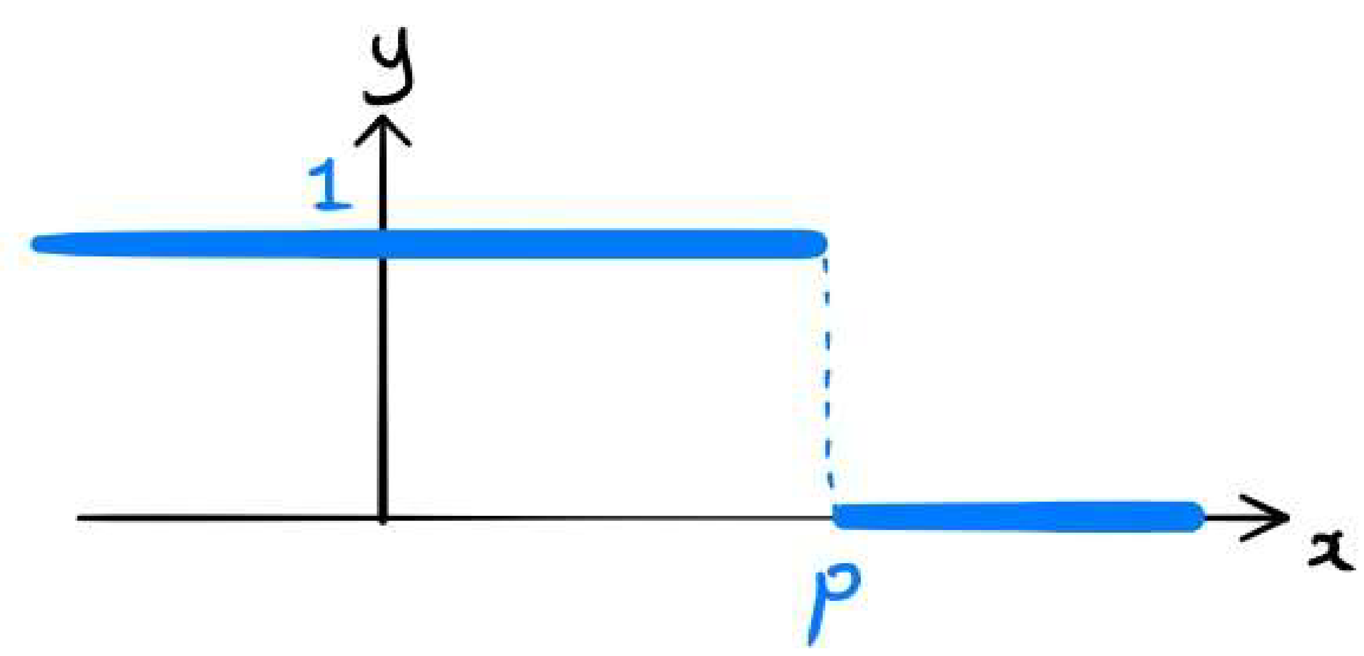
$$p_Y(y) = P(-3X+7=y) = P\left(X = \frac{7-y}{3}\right) = p_X\left(\frac{7-y}{3}\right).$$ $\square$

$\boxed{2}$ Case : X is continuous.

(1) In this case, it is not guaranteed that $Y = u(X)$ is continuous.

Ex Let $p \in [0,1]$ and

$$u(x) = \begin{cases} 1, & x < p, \\ 0, & x \geq p. \end{cases}$$



graph of $y = u(x)$

If X is $\mathcal{U}(0,1)$, then $Y = u(X)$ takes only two values $0, 1$, and

$$p_Y(0) = P(Y=0) = P(X \geq p) = 1-p,$$
$$p_Y(1) = P(Y=1) = P(X < p) = p.$$

So Y has a Bernoulli dist. w/ parameter p. $\square$

(2) If both X and $Y = u(X)$ are continuous RVs, we may relate $f_X(x)$ and $f_Y(y)$ by "CDF Technique":

▷ **CDF Technique** : Write

$$F_Y(y) = P(Y \leq y) = P(u(X) \leq y)$$
$$= P(\text{solving } u(X) \leq y \text{ in terms of } X)$$

and differentiate both sides w.r.t. y.

▷ **Example (Change-of-Variables technique)**

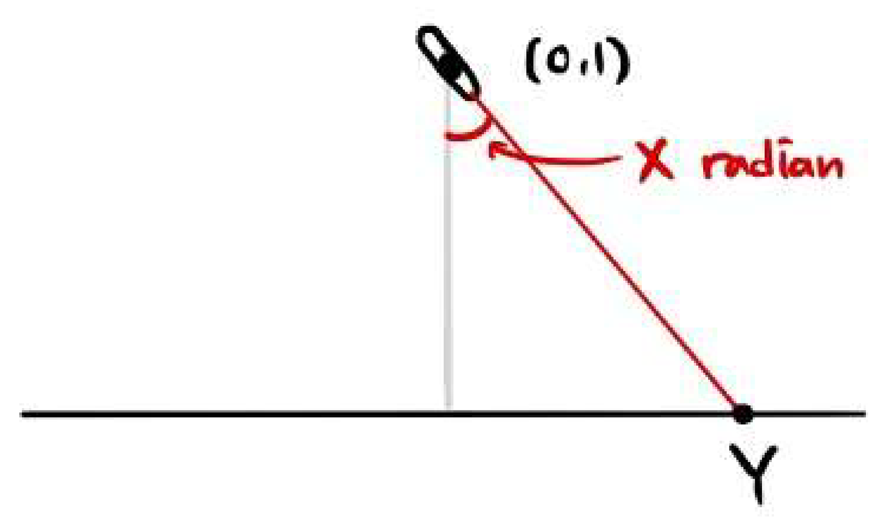
Suppose $\left\{ \begin{array}{ll} S_X : & \text{interval}, \\ u(x) : & \text{monotone} \end{array} \right\}$. Write $v = u^{-1}$: inverse fn. Then for $y \in S_Y$,

$$F_Y(y) = P(Y \le y) = P(u(X) \le y)$$
$$= \begin{cases} P(X \le v(y)) = F_X(v(y)), & \text{if } u \text{ is increasing,} \\ P(X \ge v(y)) = 1 - F_X(v(y)), & \text{if } u \text{ is decreasing.} \end{cases}$$

Differentiating w.r.t. y,

$$f_Y(y) = \left\{ \begin{matrix} f_X(v(y))\, v'(y), & u \text{ inc.} \\ -f_X(v(y))\, v'(y), & u \text{ dec.} \end{matrix} \right\} = |v'(y)|\, f_X(v(y)).$$

Ex (Cauchy distribution) Let X be $\mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2})$ and consider:



Then $Y = u(X)$ with $u(x) = \tan(x)$. So $v(y) = u^{-1}(y) = \arctan(y)$, and

- ▷ $S_Y = (-\infty, \infty)$.
- ▷ $f_Y(y) = |v'(y)|\, f_X(v(y)) = \frac{1}{1+y^2} \cdot \frac{1}{\pi} = \frac{1}{\pi(1+y^2)}$.

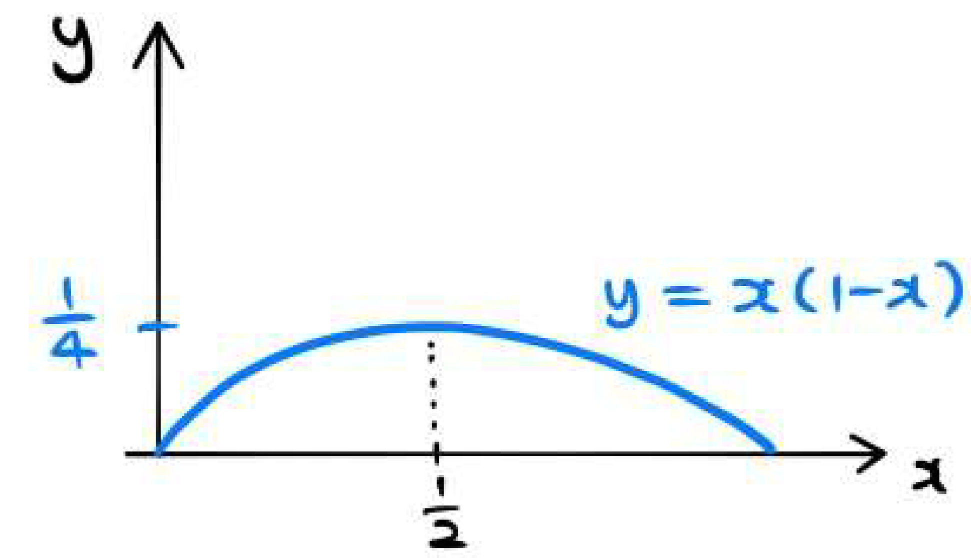
$\square$

Ex Let X has an exponential dist. w/ mean θ, and $Y = e^{-X/\theta}$. Then we may write $Y = u(X)$ with $u(x) = e^{-x/\theta}$ $\Rightarrow$ $v(y) = u^{-1}(y) = -\theta \ln y$. Then

- ▷ $S_Y = (0, 1)$,
- ▷ For each $y \in S_Y$, $f_Y(y) = |v'(y)|\, f_X(v(y)) = \left|-\frac{\theta}{y}\right| \frac{1}{\theta} e^{-(-\theta \ln y)/\theta} = 1$.

So Y has the uniform dist. $\mathcal{U}(0,1)$.

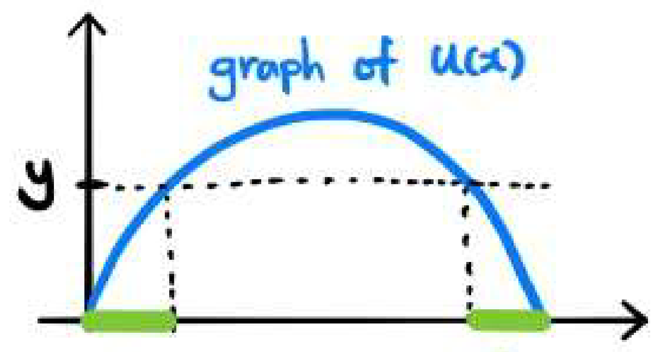
$\square$



Ex Let X be $\mathcal{U}(0,1)$ and $Y = X(1-X)$. Then

- $S_Y = [0, \frac{1}{4}]$.

- For each $y \in S_Y$,



$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(X \in \{x : u(x) \le y\}) \\ &= P\left(X \le \tfrac{1-\sqrt{1-4y}}{2}\right) + P\left(X \ge \tfrac{1+\sqrt{1-4y}}{2}\right). \\ &= F_X\left(\frac{1-\sqrt{1-4y}}{2}\right) + 1 - F_X\left(\frac{1+\sqrt{1-4y}}{2}\right). \end{aligned}$$

Differentiating,

$$f_Y(y) = \left(\frac{1}{\sqrt{1-4y}}\right)\cdot(1) + 0 - \left(-\frac{1}{\sqrt{1-4y}}\right)\cdot(1) = \frac{2}{\sqrt{1-4y}}. \qquad \square$$