Note 19

Section 4.4 Bivariate Distribution of the Continuous Type

- Extend the idea of joint dist. to two RVs of continuous type.

1 Joint PDF

DEF $f_{X,Y}(x,y)$ is called a joint PDF of two RVs X & Y if:

(a) $f_{X,Y}(x,y) \geq 0$ for any x, y.

(b) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx dy = 1$,

(c) $P((X,Y) \in A) = \iint_A f_{X,Y}(x,y)\, dx dy$ for any region A in the plane.

- As before, PDF of each X and Y is called marginal PDF and can be computed by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy, \qquad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx.$$

- CAUTION Unlike the discrete-type case, two continuous RVs need not have a joint PDF! One such class of examples are RVs X & Y related by a function, say $Y = u(X)$ for some $u(x)$. Then even if both X and Y are continuous, the joint dist. of X and Y is not "jointly continuous".

Ex Let X & Y have the joint PDF

$$f(x,y) = \frac{4}{3}(1-xy), \qquad 0<x<1, \quad 0<y<1.$$

Then

(1) $P(X<Y) = P((X,Y) \in A)$, where

$$A = \{(x,y) : x<y\},$$

and so,

$$P(X<Y) = \iint_A f(x,y)\,dx\,dy = \iint_{x<y} f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^y \frac{4}{3}(1-xy)\,dx\,dy = \int_0^1 \left[\frac{4}{3}x - \frac{2}{3}x^2 y\right]_{x=0}^{x=y} dy$$

$$= \int_0^1 \left(\frac{4}{3}y - \frac{2}{3}y^3\right) dy = \left[\frac{2}{3}y^2 - \frac{1}{6}y^4\right]_0^1 = \frac{1}{2}.$$

(2) The marginal PDF of X is, for $0<x<1$,

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \frac{4}{3}(1-xy)\,dy$$

$$= \left[\frac{4}{3}y - \frac{2}{3}xy^2\right]_{y=0}^{y=1} = \frac{4}{3} - \frac{2}{3}x, \qquad 0<x<1$$

(3) The expectation of X may be computed in two ways:

- Using the marginal PDF:

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x\cdot\left(\frac{4}{3} - \frac{2}{3}x\right) dx = \left[\frac{2}{3}x^2 - \frac{2}{9}x^3\right]_0^1 = \frac{4}{9}$$

- Using the joint PDF:

$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\,dx\,dy = \int_0^1\int_0^1 x\cdot\frac{4}{3}(1-xy)\,dx\,dy$$

$$= \int_0^1 \left[\frac{2}{3}x^2 - \frac{4}{9}x^3 y\right]_{x=0}^{x=1} dy = \int_0^1 \left(\frac{2}{3} - \frac{4}{9}y\right) dy$$

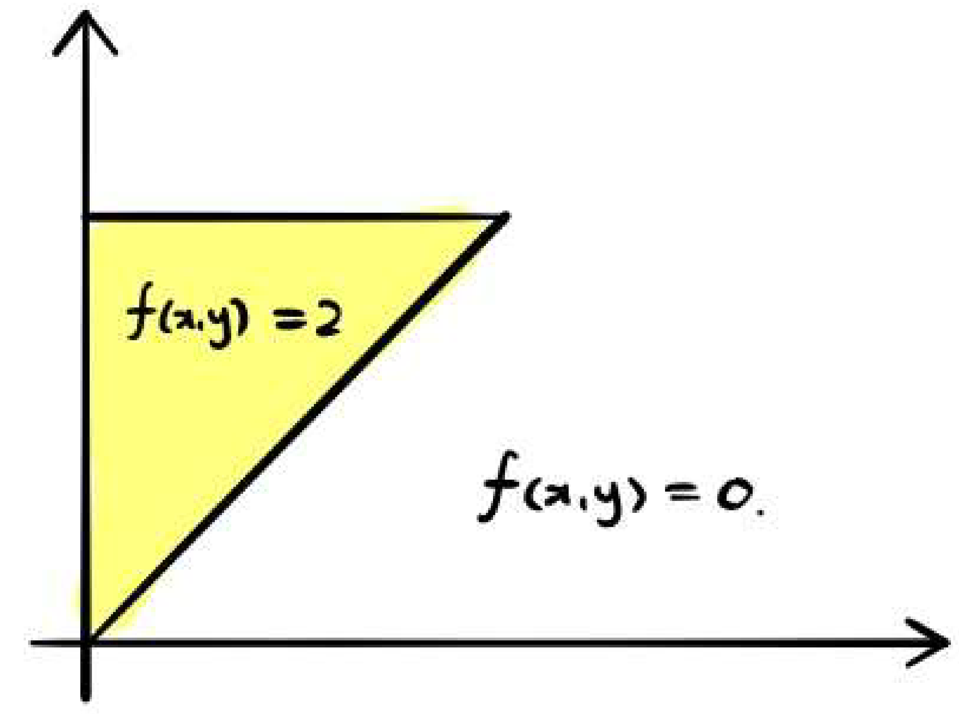
$$= \left[\frac{2}{3}y - \frac{2}{9}y^2\right]_0^1 = \frac{4}{9}.$$

□

2 Independence

DEF Let X and Y have the joint PDF $f_{X,Y}(x,y)$. X and Y are **independent** if

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) \qquad \text{for any } x, y.$$



Ex Let X and Y have the joint PDF

$$f(x,y) = 2, \qquad 0<x<y<1.$$

Then the marginal PDFs are

- If $0<x<1$, then $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_x^1 2\,dy = 2(1-x)$,

  otherwise, $f_X(x) = 0$. (why?)

- If $0<y<1$, then $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^y 2\,dx = 2y$.

  otherwise, $f_Y(y) = 0$.

- Then

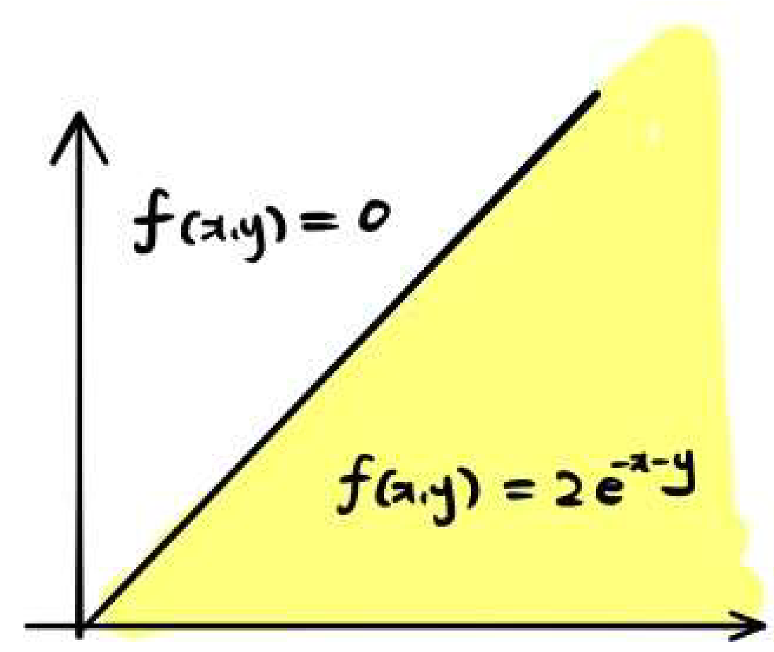
$$f_{X,Y}(x,y) = \begin{cases} 2, & 0<x<y<1, \\ 0, & \text{elsewhere,} \end{cases}$$

but

$$f_X(x)f_Y(y) = \begin{cases} 4(1-x)y, & 0<x<1,\ 0<y<1, \\ 0, & \text{elsewhere} \end{cases}$$

so X and Y are not independent. $\square$

3 Correlation Coefficient



Ex Let X and Y have the joint PDF:

$$f(x,y) = 2e^{-x-y}, \quad 0<y<x<+\infty$$

Then

$$\mu_X = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\,dx\,dy = \int_0^{\infty}\int_0^{x} x\cdot 2e^{-x-y}\,dy\,dx = \int_0^{\infty} 2x e^{-x}(1-e^{-x})\,dx = \frac{3}{2}.$$

$$\mu_Y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f(x,y)\,dx\,dy = \int_0^{\infty}\int_y^{\infty} y\cdot 2e^{-x-y}\,dx\,dy = \int_0^{\infty} 2y e^{-2y}\,dy = \frac{1}{2}.$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\,dy = \int_{0}^{\infty}\int_{y}^{\infty} 2xy\, e^{-x-y}\, dx\,dy$$

$$= \int_{0}^{\infty} 2ye^{-y}\left[-xe^{-x} - e^{-x}\right]_{x=y}^{x=+\infty} dy = \int_{0}^{\infty} 2y(y+1)\, e^{-2y}\, dy = 1.$$

So we get

$$Cov(X,Y) = E(XY) - \mu_X\mu_Y = 1 - \left(\tfrac{1}{2}\right)\cdot\left(\tfrac{3}{2}\right) = \tfrac{1}{4}.$$

In particular, X and Y are not independent. (Recall: indep. $\Rightarrow$ zero cov.) $\square$

4 Conditional Distributions

DEF Let X and Y have the joint PDF $f(x,y)$.

- The conditional PDF of Y given $X = x$ is defined by

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}, \quad \text{provided } f_X(x) > 0.$$

- The conditional expectation of $u(Y)$ given $X = x$ is defined by

$$E[u(Y) \mid X = x] = \int_{-\infty}^{\infty} u(y)\, f_{Y|X}(y|x)\, dy.$$

- $E(Y|X)$ and $Var(Y|X)$ are defined as in the discrete case.

Rmk) Both the law of total exp/var hold in this case.

Ex Let X and Y be as in the previous ex:

$$f(x,y) = 2e^{-x-y}, \quad 0 < y < x < +\infty$$

Then we can check that

$$f_X(x) = 2e^{-x}(1 - e^{-x}), \qquad 0 < x < +\infty,$$

and so,

$$f_{Y|X}(y|x) = \frac{2e^{-x-y}}{2e^{-x}(1-e^{-x})} = \frac{e^{-y}}{1-e^{-x}}, \qquad 0<y<x.$$

Then for $0<x<+\infty$,

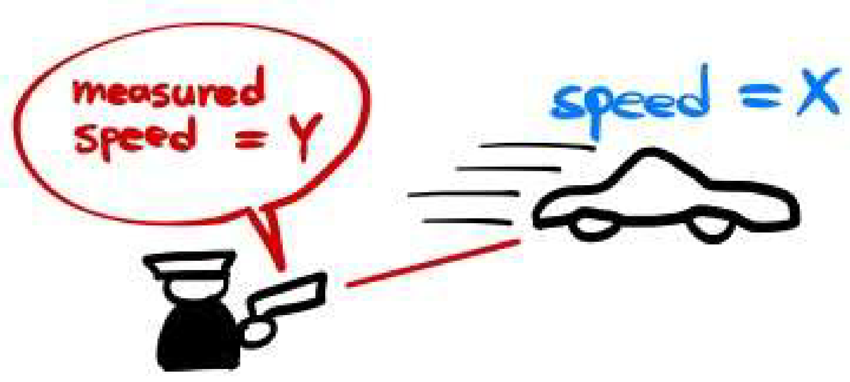
$$E(Y|X=x) = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|x)\,dy = \int_0^x \frac{ye^{-y}}{1-e^{-x}}\,dx$$

$$= \frac{1}{1-e^{-x}}\Big[-(y+1)e^{-y}\Big]_{y=0}^{y=x} = \frac{1}{1-e^{-x}}\left(1-(x+1)e^{-x}\right)$$

$$= 1 - \frac{x}{e^x-1}.$$

$\square$

Ex A police officer is measuring the speed of cars on a highway.



- The actual speed X of a car is $\mathcal{U}(80, 120)$,
- Due to the inaccuracy of the speed gun, the speed Y measured, given $X=x$, is $\mathcal{N}(x, \frac{1}{100}x)$.

Find the mean / var of Y.

Sol) By the law of total E:

$$E(Y) = E[E(Y|X)] = E[X] = 100.$$

Likewise, by the law of total Var:

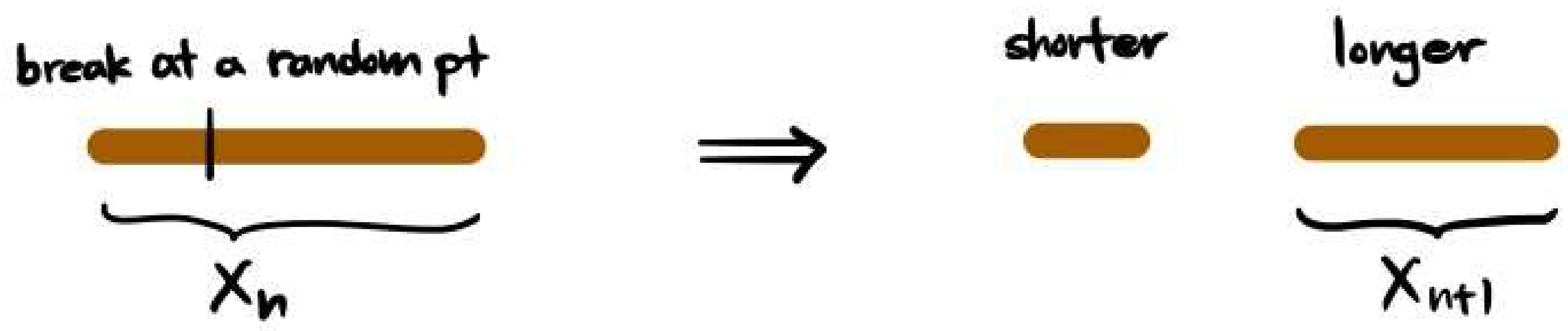
$$\begin{aligned} Var(Y) &= E(Var(Y|X)) + Var(E(Y|X)) \\ &= E\left(\tfrac{1}{100}X\right) + Var(X) \\ &= \tfrac{1}{100}\cdot 100 + \tfrac{1}{12}\cdot(120-80)^2 \\ &= 1 + \tfrac{400}{3}. \end{aligned}$$

$\square$

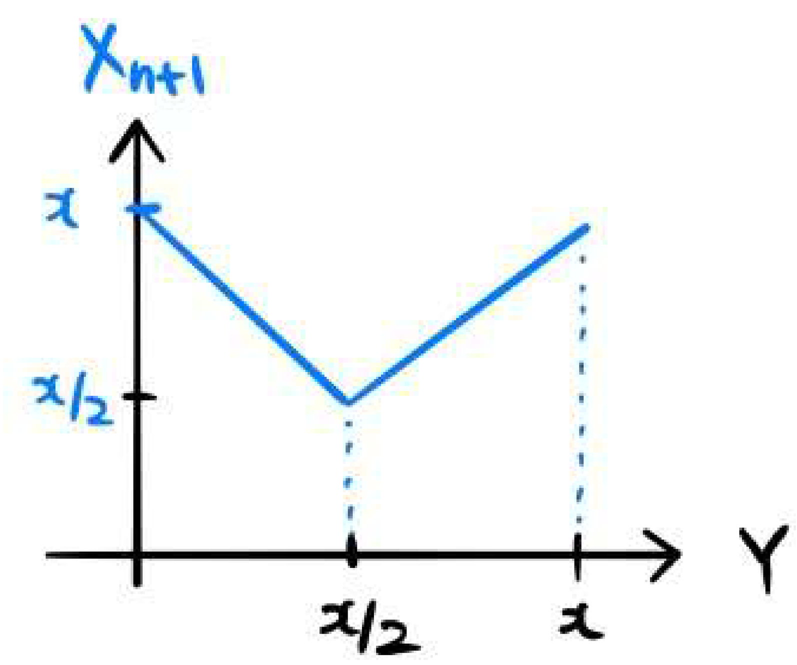
Ex Let $X_0, X_1, \cdots$ be defined as follows :

(1) Start with a stick of length X_0.

(2) If we have a stick of length X_n, break it at a uniformly randomly chosen point and X_{n+1} be the length of the longer piece :



▷ Given $X_n = x$, let Y be the breaking position. Then Y is $\mathcal{U}(0,x)$, and $X_{n+1} = \max\{Y, x-Y\}$:



So we get

$$E(X_{n+1} \mid X_n = x) = E(\max\{Y, x-Y\} \mid X_n = x)$$

$$= \int_0^x \max\{y, x-y\} \cdot \frac{1}{x}\, dy = \frac{3}{4}x,$$

and hence $E(X_{n+1} \mid X_n) = \frac{3}{4} X_n$. Then

$$E(X_n) \underset{\substack{\uparrow \\ \text{Law of Total Exp.}}}{=} E[E(X_n \mid X_{n-1})] = E\left(\frac{3}{4} X_{n-1}\right) = \cdots = \left(\frac{3}{4}\right)^n E(X_0).$$