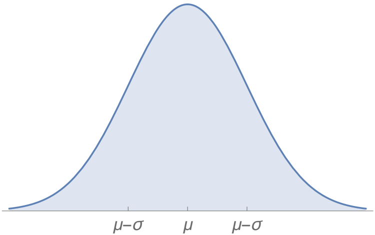
Section 3.3 The Normal Distribution

Last time, we learned:

- X has a **normal distribution** if its PDF is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

In this case, we say that X is $\mathcal{N}(\mu, \sigma^2)$.



Review

Last time, we learned:

- If X is $\mathcal{N}(\mu, \sigma^2)$, then

$$E(X) = \mu \qquad \text{and} \qquad \mathrm{Var}(X) = \sigma^2.$$

  Moreover, its MGF is given by

$$M_X(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right).$$

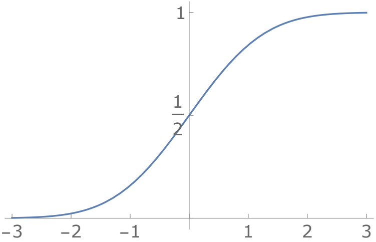
Review

Last time, we learned:

- $\mathcal{N}(0,1)$ is called a **standard normal distribution**.
- $\Phi(z)$ denotes the CDF of $\mathcal{N}(0,1)$:

$$\Phi(z) := P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, \mathrm{d}w,$$

  where Z is a standard normal variable.



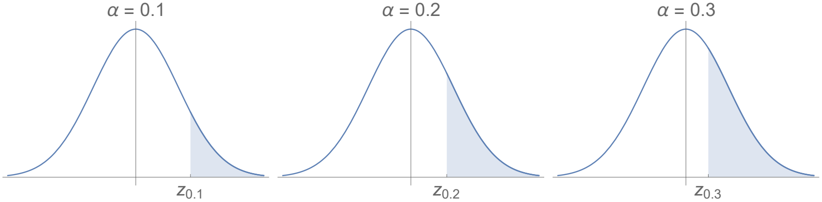
Standard normal distribution

For $0 \le \alpha \le 1$, we define z_α as the number satisfying

$$P(Z > z_\alpha) = \alpha,$$

or equivalently,

$$\Phi(z_\alpha) = 1 - \alpha.$$



Standardization

We also learned:

If X is $\mathcal{N}(\mu, \sigma^2)$, then $Z = \dfrac{X - \mu}{\sigma}$ is $\mathcal{N}(0, 1)$.

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- This Z is called the **standard score** of X, and
- Transforming X to Z is called **standardization**.

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- Transforming X to Z is called **standardization**.

We can use this to compute (or at least represent) the probabilities involving normal variables.

Standardization

Example 1

If X is $\mathcal{N}(4, 5^2)$, then

$$\begin{aligned} P(3 < X < 6) &= P\left(\frac{3-4}{5} < \frac{X-4}{5} < \frac{6-4}{5}\right) \\ &= P\left(-0.2 < Z < 0.4\right) \\ &= \Phi(0.4) - \Phi(-0.2). \end{aligned}$$

Example 2

Suppose that

$$X = [\text{volume of a certain type of canned beer}]$$

is distributed according to a normal distribution with mean $\mu = 12.1$ oz.

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is distributed according to a normal distribution with mean $\mu = 12.1$ oz.

Q. What is the value of σ so that

$$P(X > 12) = 0.99\ ?$$

Example 2

Solution.

$$\begin{aligned} P(X > 12) &= P\left(\frac{X - 12.1}{\sigma} > \frac{12 - 12.1}{\sigma}\right) \\ &= P(Z > -0.1/\sigma) \\ &= 1 - P(Z > 0.1/\sigma). \end{aligned}$$

EXAMPLE 2

Solution.

$$\begin{aligned} P(X > 12) &= P\left(\frac{X - 12.1}{\sigma} > \frac{12 - 12.1}{\sigma}\right) \\ &= P(Z > -0.1/\sigma) \\ &= 1 - P(Z > 0.1/\sigma). \end{aligned}$$

We want this to be 0.99, and so, we must have:

$$\frac{0.1}{\sigma} = z_{0.01} = 2.326.$$

This gives $\sigma = 0.043$.

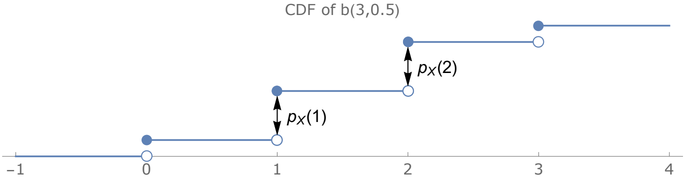
Section 3.4 Additional Models

Recall: If X is a RV of the discrete type, then

- its distribution is captured by the PMF:

$$p_X(x) = P(X = x).$$

- In the graph of the CDF, $p_X(x)$ corresponds to the jump size at x:



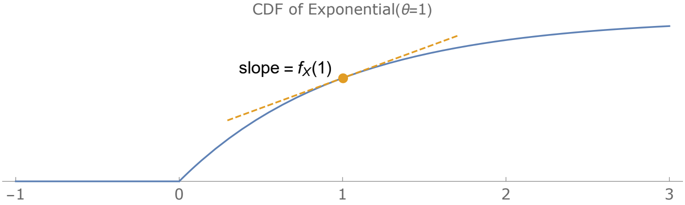
Random Variables of the mixed type

Recall: If X is a RV of the continuous type, then

- its distribution is captured by the PDF:

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} P(X \leq x)$$

- In the graph of the CDF, $f_X(x)$ corresponds to the slope at x:



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- In such cases, both the

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and

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(*Remark:* The quotation mark emphasizes that these are not truly PMF/PDF, as they do not sum/integrate up to 1.)

Random Variables of the mixed type

Example

Consider the following game: Flip a fair coin. If the outcome is:

- H $\Rightarrow$ receives \$3.
- T $\Rightarrow$ receives the amount of \$ chosen uniformly at random on $[0, 2]$.

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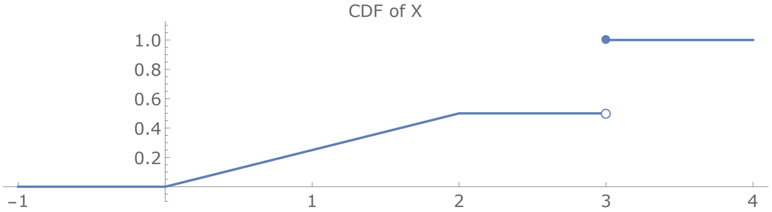
- H $\Rightarrow$ receives \$3.
- T $\Rightarrow$ receives the amount of \$ chosen uniformly at random on $[0, 2]$.

If $X =$ [amount received], then its CDF $F_X(x)$ is:



Example

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x/4, & 0 \leq x < 2, \\ 1/2, & 1 \leq x < 3, \\ 1, & 3 \leq x. \end{cases}$$

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Q. How to compute the mean and variance of X?

EXAMPLE

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Q. How to compute the mean and variance of X?

A. Both the "PDF" and "PMF" terms are needed:

$$E[u(X)] = \int_{-\infty}^{\infty} u(x) F_X'(x) \, \mathsf{d}x + \sum_x u(x) P(X = x).$$

Example

Since

$$F_X'(x) = \begin{cases} \frac{1}{4}, & 0 < x < 2, \\ 0, & \text{elsewhere} \end{cases} \quad \text{and} \quad P(X = x) = \begin{cases} \frac{1}{2}, & x = 3, \\ 0, & \text{elsewhere}, \end{cases}$$

we get

$$\begin{aligned} E[X] &= \int_0^2 x \cdot \frac{1}{4}\, \mathrm{d}x + 3 \cdot \frac{1}{2} \\ &= \left[\frac{x^2}{8}\right]_0^2 + \frac{3}{2} \\ &= 2, \end{aligned}$$

Example

and

$$\begin{aligned}\mathrm{Var}(X) &= E(X^2) - (E(X))^2 \\ &= \int_0^2 x^2 \cdot \frac{1}{4}\,\mathsf{d}x + 3^2 \cdot \frac{1}{2} - 2^2 \\ &= \left[\frac{x^3}{12}\right]_0^2 + \frac{9}{2} - 4 \\ &= \frac{7}{6}.\end{aligned}$$

Section 4.1 Bivariate Distributions of the Discrete Type

- **Q.** How to deal with two or more random quantities simultaneously?
- **Problem.** Say we have two RVs X and Y. To study how they are related, it is not sufficient to know only how each of X and Y behaves.

Motivation

- **Q.** How to deal with two or more random quantities simultaneously?

Example

Flip a fair coin, marked with 0 and 1, twice.

- **Case 1:** Let

$$X = [1^{\text{st}} \text{ outcome}], \qquad Y = [2^{\text{nd}} \text{ outcome}].$$

Both X and Y are $b(1, \frac{1}{2})$, and by the independence,

$$P(X = Y) = P(X = 0)P(Y = 0) + P(X = 1)P(Y = 1) = \frac{1}{2}.$$

- **Q.** How to deal with two or more random quantities simultaneously?

Example

Flip a fair coin, marked with 0 and 1, twice.

- **Case 2:** Now let

$$X = [1^{\text{st}} \text{ outcome}] = Y.$$

Again, both X and Y are $b(1, \frac{1}{2})$. However,

$$P(X = Y) = 1.$$

Motivation

- **Q.** How to deal with two or more random quantities simultaneously?
- **Solution.** Keep track of the information “jointly”.

Definition

Let X and Y be RVs of the discrete type. Then the **joint probability mass function** (joint PMF) is the function

$$p(x, y) = p_{X,Y}(x, y) = P(\underbrace{X = x,\ Y = y}_{X=x \text{ and } Y=y}).$$

Properties of joint PMF

Let X and Y be RVs of the discrete type. Let

$$S = [\text{space of } (X,Y)],$$

i.e., the set of all possible values of the pair (X,Y).

Properties of joint PMF

Let X and Y be RVs of the discrete type. Let

$$S = [\text{space of } (X, Y)],$$

i.e., the set of all possible values of the pair (X, Y).Then the joint PMF $p(x, y)$ of X and Y satisfies:

1. $0 \leq p(x, y) \leq 1$, for any real x, y.
2. $\displaystyle\sum_{(x,y)\in S} p(x, y) = 1.$
3. $\displaystyle P((X, Y) \in A) = \sum_{(x,y)\in A} p(x, y)$, for any $A \subseteq S$.

Example

Roll a pair of fair dice and let

$$X = [\text{smaller outcome}], \qquad Y = [\text{larger outcome}].$$

Then the space S is

$$S = \{(x, y) : x, y \text{ integers satisfying } 1 \leq x \leq y \leq 6\}.$$

Also, for integers x and y,

$$p(x, y) = \begin{cases} P(\text{outcome is } (x, y) \text{ or } (y, x)) = \frac{2}{36}, & \text{if } 1 \leq x < y \leq 6, \\ P(\text{outcome is } (x, x)) = \frac{1}{36}, & \text{if } 1 \leq x = y \leq 6. \end{cases}$$

Bivariate Distribution

Definition

In the context of bivariate distribution, the PMF of each of X and Y is called **marginal**.

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Example

The marginal PMF $p_X(x)$ of X is simply the PMF of X. Using the joint PMF, it can be computed by

$$p_X(x) = P(X = x) = \sum_y p_{X,Y}(x, y).$$

Bivariate Distribution

Definition

X and Y are called **independent** if

$$P(X = x,\ Y = y) = P(X = x)P(Y = y)$$

for any x, y.

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Otherwise, X and Y are called **dependent**.

Example: Dependent Case

Let X and Y have the joint PMF:

$$p(x,y) = \begin{cases} \dfrac{x+y}{32}, & x = 1,2,\ y = 1,2,3,4 \\ 0, & \text{elsewhere.} \end{cases}$$

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Then for $x = 1,2$,

$$p_X(x) = \sum_{y=1}^{4} p(x,y) = \sum_{y=1}^{4} \frac{x+y}{32} = \frac{4x+10}{32}$$

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and similarly, for $y = 1,2,3,4$,

$$p_Y(y) = \sum_{x=1}^{2} p(x,y) = \sum_{x=1}^{2} \frac{x+y}{32} = \frac{2y+3}{32}.$$

EXAMPLE: DEPENDENT CASE

Since

$$p_X(x)p_Y(y) = \frac{(4x+10)(2y+3)}{32^2} \neq \frac{x+y}{32} = p_{X,Y}(x,y)$$

for any of $x = 1, 2$ and $y = 1, 2, 3, 4$, we conclude that X and Y are dependent.

EXAMPLE: INDEPENDENT CASE

Let X and Y have the joint PMF:

$$p(x, y) = \begin{cases} \dfrac{2}{2^x 3^y}, & x, y : \text{positive integers} \\ 0, & \text{elsewhere.} \end{cases}$$

Then for $x = 1, 2, 3, \cdots$,

$$p_X(x) = \sum_{y=1}^{\infty} p(x, y) = \sum_{y=1}^{\infty} \frac{2}{2^x 3^y} = \frac{1}{2^x},$$

and similarly, for $y = 1, 2, 3, \cdots$,

$$p_Y(y) = \sum_{x=1}^{\infty} p(x, y) = \sum_{x=1}^{\infty} \frac{2}{2^x 3^y} = \frac{2}{3^y}.$$

EXAMPLE: INDEPENDENT CASE

Since

$$p_X(x)p_Y(y) = \frac{2}{2^x 3^y} = p_{X,Y}(x,y)$$

for any positive integers x and y, we conclude that X and Y are independent.

Bivariate Distribution

Definition

Let X and Y have the joint PMF $p(x, y)$ and $u(x, y)$ be a function of two variables. Then $u(X, Y)$ is another random variable. Its **expected value** is computed by

$$E[u(X, Y)] = \sum_{(x,y)\in S} u(x, y)p(x, y).$$

Bivariate Distribution

Example

If X and Y have the joint PMF $p(x, y)$, then the expectation of X can be computed by

$$E[X] = \sum_{(x,y)\in S} xp(x, y)$$

using the function $u(x, y) = x$, and likewise, the expectation of Y can be computed by

$$E[Y] = \sum_{(x,y)\in S} yp(x, y)$$

using the function $u(x, y) = y$.

Example

Flip a fair coin, marked with 0 and 1, twice. Let

$$X_1 = [1^{\text{st}} \text{ outcome}], \qquad X_2 = [2^{\text{nd}} \text{ outcome}].$$

Then the joint PMF is

$$p(x_1, x_2) = \frac{1}{4}, \qquad x_1 = 0, 1 \quad \text{and} \quad x_2 = 0, 1.$$

Now consider $Y = (X_1 - X_2)^2$. Then

$$\begin{aligned} E[Y] = E[(X_1 - X_2)^2] &= \sum_{x_1=0}^{1} \sum_{x_2=0}^{1} (x_1 - x_2)^2 \cdot \frac{1}{4} \\ &= (0-0)^2 \frac{1}{4} + (0-1)^2 \frac{1}{4} + (1-0)^2 \frac{1}{4} + (1-1)^2 \frac{1}{4} = \boxed{\frac{1}{2}}. \end{aligned}$$