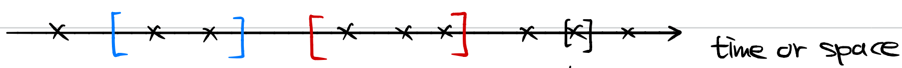
Note 13

Section 3.2. The Exponential, Gamma, and Chi-Square Distributions

① Exponential distribution.

- Recall the approx. Poisson process, where



① # of occurrences in disjoint intervals are indep,

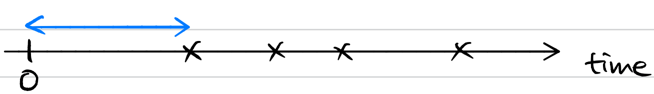
② # of occurrences in a small interval of length h is almost Bernoulli w/ param λh.

Under this assumption,

Poisson random variable = [# of occurrences on a given interval.]

- Now, we ask:

Q How long do we have to wait until the first occurrence?



Let W = [waiting time until the first occurrence]

- Then W is a non-negative, continuous RV. To find the distribution, we investigate CDF:

For $w \geq 0$,
$$\begin{aligned} F(w) &= P(W \leq w) \\ &= 1 - P(W > w) \\ &= 1 - P(\underbrace{\text{no occurrence in } [0, w]}_{\text{can be computed using Poisson distribution!}}) \\ &= 1 - e^{-\lambda w}. \end{aligned}$$

- PDF is the derivative of CDF:

$$f(w) = F'(w) = \lambda e^{-\lambda w}, \qquad w \geq 0.$$

DEF A RV X has an exponential distribution with mean $\theta = 1/\lambda$ if it has PDF:

$$f(x) = \frac{1}{\theta} e^{-x/\theta}, \qquad x \geq 0.$$

PROP If X has exp. dist. w/ mean θ, then

$$E(X) = \theta, \quad Var(X) = \theta^2, \quad M_X(t) = \frac{1}{1-\theta t}.$$

Pf)
$$M_X(t) = \int_0^\infty e^{tx} f(x)\,dx = \int_0^\infty e^{tx} \cdot \frac{1}{\theta} e^{-x/\theta}\,dx$$

$$= \left[\frac{1}{\theta} \cdot \frac{1}{t - 1/\theta} e^{(t - 1/\theta)x} \right]_{x=0}^{\infty}$$

If $t < 1/\theta$, then

$$= 0 - \frac{1}{\theta} \cdot \frac{1}{t - 1/\theta} \cdot e^0 = \frac{1}{1 - \theta t}.$$

Then

$$M'(t) = \frac{\theta}{(1-\theta t)^2}, \qquad M''(t) = \frac{2\theta^2}{(1-\theta t)^3},$$

and so,

$$E(X) = \theta, \qquad Var(X) = (2\theta^2) - \theta^2 = \theta^2. \qquad \square$$

Ex Suppose a certain type of light bulb has an exp. dist. w/ mean life of 500 hours.

Let X : life time of a light bulb. Then find:
(a) $P(X < 750)$, (b) $P(X > 900 \mid X > 300)$.

Sol) (a)
$$P(X < 750) = \int_0^{750} \frac{1}{500} e^{-x/500}\, dx$$
$$= \left[e^{-x/500} \right]_0^{750} = 1 - e^{-1.5},$$

or simply use the fact $P(X<750) = F(750)$.

(b) We have

$$P(X>900 \mid X>300) = \frac{P(\{X>900\} \cap \{X>300\})}{P(X>300)}.$$

Since $X > 900$ always implies $X > 300$,

$$= \frac{P(X>900)}{P(X>300)}$$

$$= \frac{e^{-900/500}}{e^{-300/500}} = e^{-6/5}.$$

Note This coincides with $P(X > 600) = e^{-600/500}$.

In general, if X has exp. dist. and $x, y > 0$,

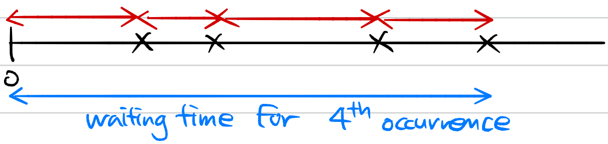
$$P(X > x+y \mid X > x) = P(X > y).$$

I.e. given that $X > x$, the "extra waiting time $X - x$" behaves the same as exp. dist. of the same mean. This is called "no-memory" property.

② Gamma distribution, special case.

<u>Q</u> How long does it takes until the α^{th} occurrence?

<u>Ex</u>



- Write W = [waiting time until α^{th} occurrence]. Then for $w > 0$, its PDF is

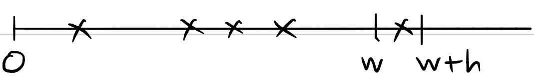
$$f(w) \approx \frac{P(w \le W \le w+h)}{h}$$

h small

$$= \frac{1}{h} P\left(\alpha^{th} \text{ occurrence is in } [w, w+h]\right)$$

$$\approx \frac{1}{h} P\left(\left\{\begin{matrix}(\alpha-1) \text{ occurrences} \\ \text{before time } w\end{matrix}\right\} \cap \left\{\begin{matrix}1 \text{ occurrence} \\ \text{in } [w, w+h]\end{matrix}\right\}\right)$$

<u>Ex</u> $\left\{\begin{matrix}5^{th} \text{ occurrence is} \\ \text{in } [w, w+h]\end{matrix}\right\}$ "$\approx$" $\Bigg\{$



$\Bigg\}$

$$= \frac{1}{h} \cdot \frac{(\lambda w)^{\alpha-1}}{(\alpha-1)!} e^{-\lambda w} \cdot \frac{(\lambda h)^1}{1!} e^{-\lambda h}$$

$$\approx \frac{\lambda^\alpha w^{\alpha-1}}{(\alpha-1)!} e^{-\lambda w}$$

$$= \frac{w^{\alpha-1}}{(\alpha-1)!\,\theta^\alpha} e^{-w/\theta}. \qquad (\theta = \tfrac{1}{\lambda})$$

As $h \to 0^+$, this becomes an identity. We say W has a gamma distribution.

③ Gamma distribution, general case

- The above formula allows a generalization.

DEF The gamma function is defined by

$$\Gamma(t) = \int_0^\infty y^{t-1} e^{-y}\, dy, \qquad t > 0.$$

PROP (a) $\Gamma(1) = 1$,

(b) $\Gamma(t) = (t-1)\Gamma(t-1)$, for $t > 1$.

(c) $\Gamma(n) = (n-1)!$, for $n = 1, 2, 3, \cdots$.

I.e., Γ generalizes factorial.

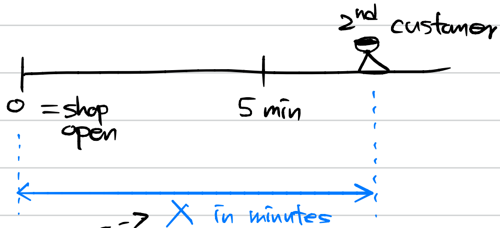
- Using this, we define:

DEF X has a gamma distribution if its PDF is

$$f(x) = \frac{1}{\Gamma(\alpha)\,\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, \qquad x \geq 0,\ \alpha > 0,\ \theta > 0.$$

Rmk) Of course, $\alpha = 1$ case reduces to exp. dist.

Ex Suppose [# of 's / hour arriving at a shop] follows Poisson process w/ mean 20. I.e., $\theta = 3$ mins / .
What is

$P($  $) = ?$

Sol) Let X as $\dashrightarrow$ X in minutes. Then X has a gamma dist. w/ $\alpha = 2$, $\theta = 3$. So

$$P(\sim) = P(X > 5) = \int_5^\infty \frac{1}{\Gamma(2) \cdot 3^2} x^{2-1} e^{-x/3}\, dx$$

$$= \int_5^\infty \frac{x e^{-x/3}}{9}\, dx$$

$$= \frac{1}{9}\Big[(-3) x e^{-x/3} - 9 e^{-x/3} \Big]_5^\infty$$

$$= \frac{8}{3} e^{-5/3}.$$

Alternatively,

$$P(\sim) = P(\text{either } 0 \text{ or } 1 \text{ up to } 5 \text{ mins})$$

$$= \sum_{k=0}^{\alpha - 1} \frac{(5/3)^k}{k!} e^{-5/3} = \frac{8}{3} e^{-5/3}.$$

[# of 's up to 5 mins] has Poisson dist. w/ rate 5/3

Converting to Poisson process is often easier.

PROP If X has gamma dist., then

$$E(X) = \alpha\theta, \quad Var(X) = \alpha\theta^2, \quad M(t) = \frac{1}{(1-\theta t)^\alpha}.$$

④ Chi-Square Distribution

Q What is the use of gamma dist w/ non-integer α?

A Some other distributions, arising in different contexts, reduce to gamma dist.

DEF X has chi-square distribution w/ r degree of freedom if it has a gamma dist. w/ $\theta = 2$ and $\alpha = \frac{r}{2}$.
I.e.,

$$f(x) = \frac{1}{\Gamma(r/2)\, 2^{r/2}} x^{\frac{r}{2}-1} e^{-x/2}, \qquad x > 0.$$

This is often abbreviated by saying:

$$X \sim \chi^2(r).$$

Greek letter χ (chi)

- So,

$$E(X) = 2 \cdot \frac{r}{2} = r, \qquad Var(X) = 2^2 \cdot \frac{r}{2} = 2r.$$

- $\chi^2_\alpha(r) :=$ [$100(1-\alpha)^{th}$ percentile].