Note 12

3.1. RV of the continuous type

$\boxed{1}$ • Let X = pos. of a point chosen "uniformly" at random from $[a, b]$.

$\Rightarrow$ For $a \le x \le b$, the CDF

$$F(x) = P(X \le x)$$

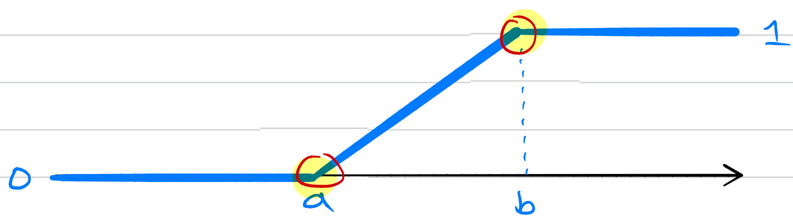
is proportional to the length of $[a, x]$, so

$$F(x) = \frac{x - a}{b - a}.$$

normalizes total prob. to 1.

$\Rightarrow$ For general $x \in \mathbb{R}$,

$$F(x) = \begin{cases} 0, & x < a, \\ \dfrac{x-a}{b-a}, & a \le x < b, \\ 1, & b \le x. \end{cases}$$



• We may write:

$$F(x) = \int_{-\infty}^{x} f(y)\,dy$$

for

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b, \\ 0, & \text{elsewhere} \end{cases}$$

exact value of f at a & b is not important, b.c. F is not diff-ble at a & b.

Alternatively, $F'(x) = f(x)$.

- This $f(x)$ is called the probability density function (PDF) of X. This is an example of:

(a.k.a. continuous RV)

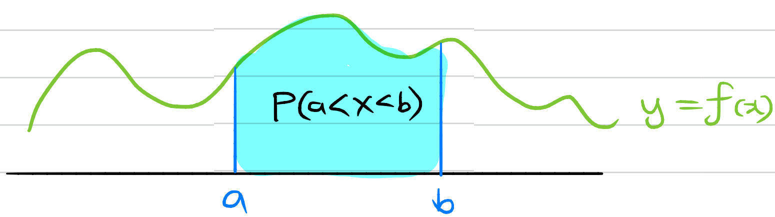
DEF The PDF of a RV X of the continuous type, with space $=$ (union of) interval(s), is an integrable ftn s.t.

for all

(a) $f(x) \geq 0, \quad \forall x \in S.$

(b) $\int_S f(x)\,dx = 1.$

(c) $P(a<X<b) = \int_a^b f(x)\,dx, \quad \forall (a,b) \subseteq S.$



Rmk 1) Can always extend f to $\mathbb{R}$ by setting

$$\text{"new } f(x)\text{"} = \begin{cases} f(x), & x \in S \\ 0, & \text{elsewhere.} \end{cases}$$

Then (a) – (c) becomes:

(a') $f(x) \geq 0, \quad \forall x \in \mathbb{R}$

(b') $\int_{-\infty}^{\infty} f(x)\,dx = 1.$

(c) $P(a<X<b) = \int_a^b f(x)\,dx, \quad \forall (a,b) \subseteq \mathbb{R}.$

<u>Rmk 2)</u> PMF $f(x)$ is NOT a probability. In particular, a PMF can exceed 1.

<u>Rmk 3)</u> If X is a conti. RV with PMF f, then

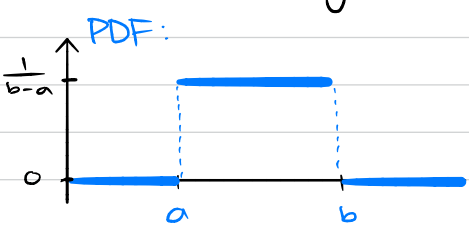
$$P(X=x) = \int_x^x f(y)\,dy = 0.$$

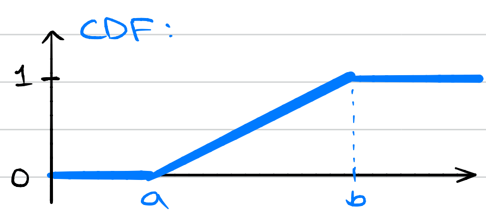
So, $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a<X<b)$.

<u>Ex</u> (Uniform distribution) Both descriptions

$$\left[f(x) = \frac{1}{b-a}, \quad a<x<b \right] \quad \text{and} \quad \left[f(x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & e/w \end{cases} \right]$$

refer to the PDF of the same dist., called the uniform distribution (or rectangular dist.), which is denoted by $U(a,b)$.





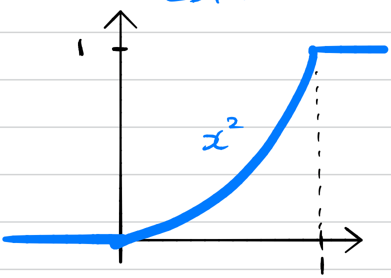
<u>Ex</u> Let X has CDF

$$F(x) = \begin{cases} 0, & x < 0, \\ x^2, & 0 \le x < 1, \\ 1, & 1 \le x. \end{cases}$$ $\square$

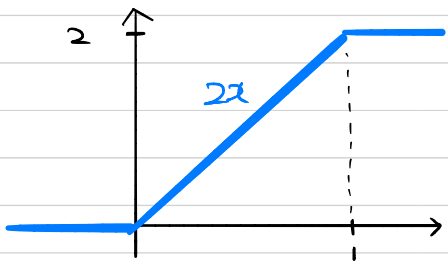
Then its PDF is

$$f(x) = F'(x) = \begin{cases} 2x, & 0 < x < 1, \\ 0, & \text{elsewhere}. \end{cases}$$

CDF:



PDF:



Now,

$$P\left(\tfrac{1}{3} < X < \tfrac{2}{3}\right) = F\left(\tfrac{2}{3}\right) - F\left(\tfrac{1}{3}\right) = \left(\tfrac{2}{3}\right)^2 - \left(\tfrac{1}{3}\right)^2 = \frac{1}{3},$$

and

$$P\left(\tfrac{1}{2} < X < 2\right) = F(2) - F\left(\tfrac{1}{2}\right) = 1 - \left(\tfrac{1}{2}\right)^2 = \frac{3}{4}.$$

Alternatively,

$$P\left(\tfrac{1}{2} < X < 2\right) = \int_{\frac{1}{2}}^{2} f(x)\,dx = \int_{\frac{1}{2}}^{1} 2x\,dx + \int_{1}^{2} 0\,dx$$

$$= \frac{3}{4}.$$ $\square$

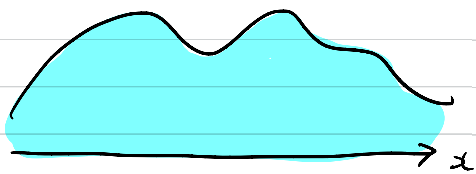
Often, PDF is piecewise-defined.

2 Expectation

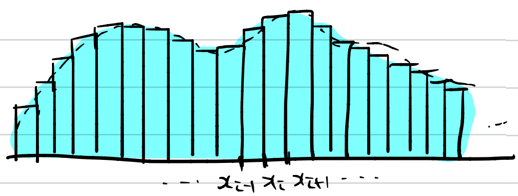
DEF If X : conti. RV with PDF $f(x)$, then its **expected value** (or **mean**) is

$$\mu = E(X) := \int_{-\infty}^{\infty} x f(x)\, dx.$$

Motivation



$\approx$

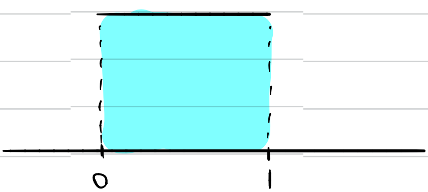


graph of $f(x)$

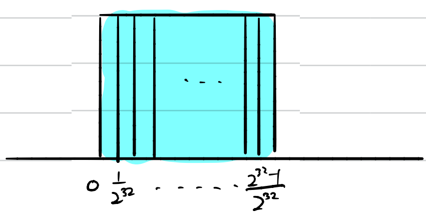
prob. histogram of
$p(x_i) = P(x_i \le X < x_{i+1})$
$\approx f(x_i)\, \Delta x$
with bin size Δx.

$$\Rightarrow \int_a^b u(x) f(x)\, dx = E[u(X)] \approx \sum_{i:\, a < x_i < b} u(x_i)\, f(x_i)\, \Delta x_i.$$

(For instance in software, pseudo-random number gen. uses this idea to approx the uniform dist. $U(0,1)$ by a discrete uniform dist.:)



$\approx$



DEF • Variance : $\sigma^2 = Var(X) = E[(X-\mu)^2]$

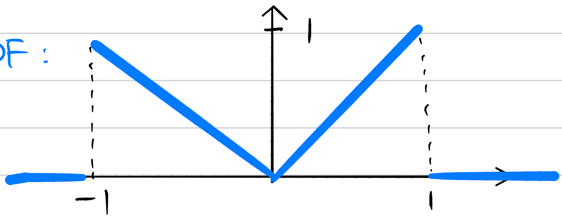
$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx,$$

• Std. Dev. : $\sigma = \sqrt{Var(X)}$,

• MGF : $M(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx.$

Ex Let X has PDF $f(x) = |x|$, $-1 < x < 1$.

PDF :



Then

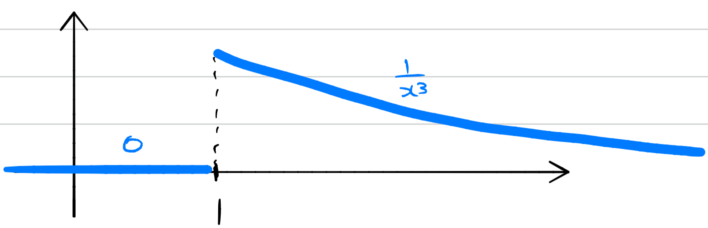
$$E(X) = \int_{-1}^{1} x \cdot |x|\, dx = \int_{-1}^{0} (-x^2)\, dx + \int_{0}^{1} x^2\, dx$$

$$= \left[-\frac{x^3}{3}\right]_{-1}^{0} + \left[\frac{x^3}{3}\right]_{0}^{1} = -\frac{1}{3} + \frac{1}{3} = 0.$$

$$Var(X) \underset{(\because \mu=0)}{=} E(X^2) = \int_{-1}^{1} x^2 |x|\, dx$$

$$= \int_{-1}^{0} (-x^3)\, dx + \int_{0}^{1} x^3\, dx = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}. \quad \square$$

Ex Let X has PDF $f(x) = \frac{2}{x^3}$, $x > 1$.



Then

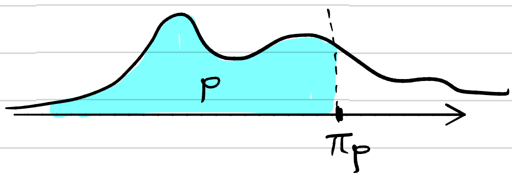
$$E(X) = \int_1^\infty x \cdot \frac{2}{x^3}\,dx = \lim_{b\to\infty} \int_1^b \frac{2}{x^2}\,dx$$

$$= \lim_{b\to\infty} \left[-\frac{2}{x} \right]_1^b = \lim_{b\to\infty} \left(2 - \frac{2}{b} \right)$$

$$= 2. \qquad \square$$

DEF The $(100p)$ quantile is a number π_p s.t.

$$F(\pi_p) = p$$



- Special cases:

p	name of π_p
0.25	first quartile
0.5	median (= second quartile)
0.75	third quartile

Rmk) Median need not equal the mean.