Note 11

Today, we continue the study some common distributions.

2.6. The Negative Binomial Distribution

- The negative binomial distribution concerns the # of independent Bernoulli trials until exactly r successes occur.

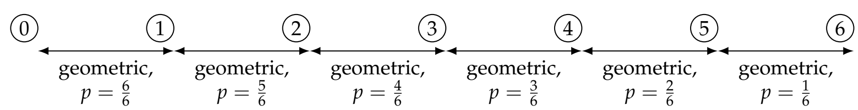
Example (Couple Collector Problem)

Each box of a brand of cereals contains a coupon, and there are 6 different types of coupons. What is the expected number of boxes to be purchased in order to collect all 6 types?

Solution. Let $i \in \{1, \cdots, 6\}$. Then, after $i-1$ different types of coupons have been collected, the probability that each box of serial contains a coupon of uncollected types is

$$p = 1 - \frac{i-1}{6} = \frac{7-i}{6}.$$

So the expected number of boxes of cereals to be purchased until a new type of coupon is collected is the mean of a geometric random variable with this $p = (7-i)/6$, which is $1/p = 6/(7-i)$.



So the expected number of boxes to be purchased in order to collect all 6 types is:

$$\frac{6}{6} + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1} = 14.7$$

□

Example

Flip a fair coin until the second head is observed, and let X be the number of total coin flips. Find the value of $P(X \le 10)$.

Solution. Note that X has a negative binomial distribution with $r = 2$ and $p = \frac{1}{2}$. So, in principle, we may find the probability by setting up the sum

$$P(X \le 10) = \sum_{x=2}^{10} \binom{x-1}{1} p^2 (1-p)^{x-2}$$

and compute this. But there is a much simpler way to solve this. The condition $X \leq 10$ is the same as saying that there are at least 2 successes in the first 10 independent Bernoulli trials with $p = \frac{1}{2}$. So if Y denotes the number of successes in the first first 10 trials, then

$$P(X \leq 10) = P(Y \geq 2) = 1 - P(Y < 2).$$

Since

$$P(Y < 2) = \sum_{y=0}^{1} \binom{10}{y} p^y (1-p)^{10-y} = \frac{11}{1024},$$

it follows that

$$P(X \leq 10) = \frac{1013}{1024} \approx 0.989\ldots$$

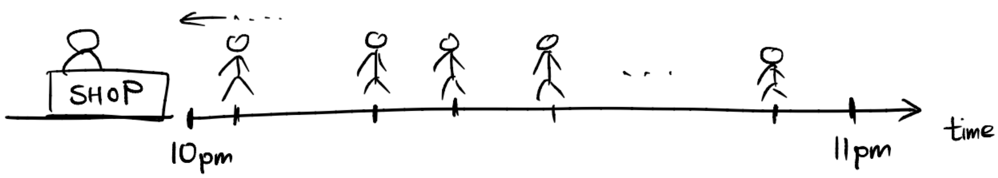
$\square$

Section 2.7. The Poisson Distribution

Approximate Poisson process

Example 1

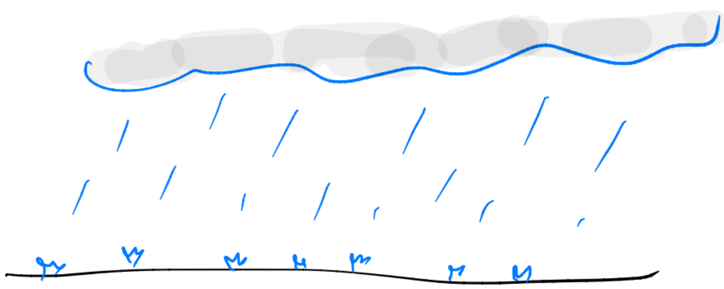
Customers visit a store. How many customer will visit between 10–11pm?



1. Each customer visit the store independently at any time they like.
2. The probability that a customer visits in a 1-second interval is small.
3. The probability that two or more customers visit in a 1-second interval is much smaller.

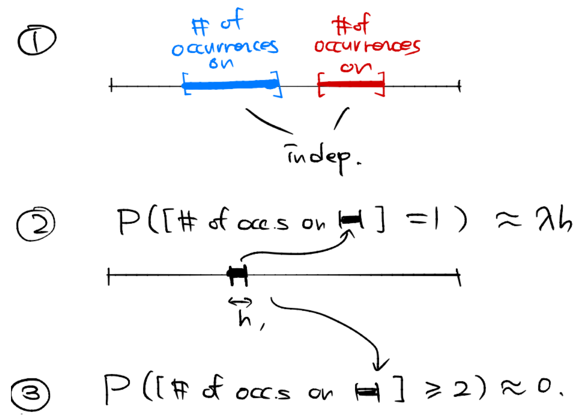
Example 2

A rain cloud has just formed and begins to pour raindrops. How many raindrops will hit the street in the first 1 minute?



1. When and where a raindrop will fall is independent.
2. The probability that a 1-sqaure-inch of area is hit by a raindrop is small.
3. The probability that the above area is hit by 2 or more raindrops is much smaller.

The common theme in each of examples can be summarized as follows. Suppose we want to count the number of occurrences of some event in a given continuous interval. Then we have an **approximate Poisson process** with parameter $\lambda > 0$ if:



Poisson distibution - unit length case

Under this process, if an interval of unit length is divided into n subintervals of equal lengths, then counting the number of occurrences on each subinterval is approximately a Bernoulli trial with parameter $p = \lambda/n$. So the total number X of occurrences may be approximated by the binomial distribution $b(n, p)$. Then for each non-negative integer x, we have:

$$\begin{aligned}
P(X = x) &\approx \binom{n}{x} p^x (1-p)^{n-x} \\
&= \frac{n(n-1)\cdots(n-x+1)}{x!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x} \\
&= \frac{\lambda^x}{x!} \left(1 - \frac{\lambda}{n}\right)^n \frac{n(n-1)\cdots(n-x+1)}{n^x} \left(1 - \frac{\lambda}{n}\right)^{-x} \\
&\approx \frac{\lambda^x}{x!} e^{-\lambda} \cdot 1 \cdot 1.
\end{aligned}$$

This approximation becomes more precise as n grows and eventually an exact identity as $n \to \infty$. We summarize:

Definition and Properties

We say that a random variable X has a **Poisson distribution** with parameter λ if it has PMF:

$$p(x) = P(X = x) = \frac{\lambda^x}{x!} e^{-\lambda}, \qquad x = 0, 1, 2, \cdots.$$

Then we have:

Mean	λ
Variance	λ
MGF	$e^{\lambda(e^t - 1)}$

Although it requires a bit of justification, both the mean and the variance can be easily guessed from the binomial approximation:

$$E(X) \approx np = \lambda, \qquad \text{Var}(X) \approx npq = \lambda\left(1 - \frac{\lambda}{n}\right) \approx \lambda.$$

The hand-waving part "≈" will become more accurate as $n \to \infty$. Of course, we can show everything using the PMF.

Proof. Recall that $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Then

$$\sum_{x=0}^{\infty} p(x) \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} e^{-\lambda} = e^{\lambda} e^{-\lambda} = 1,$$

and so, $p(x)$ is indeed a PMF. Next, its MGF is

$$M(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} p(x) = \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x}{x!} e^{-\lambda} = \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} e^{-\lambda} = e^{\lambda e^t - \lambda} = e^{\lambda(e^t - 1)}.$$

Then we have

$$E(X) = M'(0) = \lambda e^t e^{\lambda(e^t-1)}\Big|_{t=0} = \lambda$$

and

$$E(X^2) = M''(0) = \lambda e^t e^{\lambda(e^t-1)} + (\lambda e^t)^2 e^{\lambda(e^t-1)}\Big|_{t=0} = \lambda + \lambda^2.$$

So we get $\text{Var}(X) = E(X^2) - (E(X))^2 = \lambda$. □

Example 1 revisited.

Customers visit a store on the average of 3 people every hour. Let X be the number of customers between 10–11pm. Then X has a Poisson distribution with $\lambda = 3$.

- The probability that no customer visits between 10–11pm is

$$P(X = 0) = e^{-3} \approx 0.0498.$$

- The probability that 5 or more customer visits between 10–11pm is

$$P(X \geq 5) = 1 - P(X < 5) = 1 - \sum_{x=0}^{4} \frac{3^x}{x!} e^{-3} \approx 1 - 0.815 = 0.185.$$

Poisson distibution - general case

Assuming approximate Poisson process with parameter $\lambda > 0$, but now working on an interval of length t units instead, then the total number of occurrence has the Poisson distribution with parameter λt, i.e., has the PDF

$$\frac{(\lambda t)^x}{x!}e^{-\lambda t}, \qquad x = 0, 1, 2, \cdots .$$

So the meaning of λ is the "average rate of occurrence", or simply, the **rate**.

Example

In a large city, 911 calls come on the average of 2 every 3 minutes. Assuming an approximate Poisson process, what is the probability of 5 or less calls arriving in a 12-minute period? Let X denote the number of calls in the 9-minute period. Then

- The rate is 2 calls per 3 minutes,
- 12 minutes is 4 times the unit length (=3 minutes).

So X has the Poisson distrbution with parameter $2 \times 4 = 8$, and

$$P(X \leq 5) = \sum_{x=0}^{5} \frac{8^x}{x!}e^{-8} \approx 0.191.$$

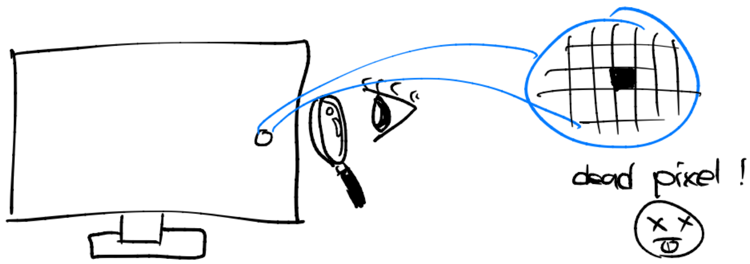
Poisson approximation

The above discussion can be used to approximate binomial distributions where the sample size is large and the success probability is small:

If n large & p small, [binomial w/ params n & p $\approx$ [Poisson with param np].

Example

You have just bought a brand-new OLED TV and you want to make sure there is no defect. There are 8,000,000 pixels in this TV, and it is known that there is 1 out of 1,000,000 pixels is dead on the average. Find the probability that your TV has 8 or more dead pixels, using Poisson approximation.



Let X be the number of dead pixels in your TV. Then X is binomially distributed with parameters $n = 8,000,000$ and $p = \frac{1}{1,000,000}$. Using Poisson approximation, X is approximately Poisson with parameter $np = 8$. So

$$P(X \geq 8) \approx 1 - \sum_{x=0}^{7} \frac{8^x}{x!} e^{-8} \approx 1 - 0.453 = 0.547.$$