Section 2.2. Mathematical Expectation

$\boxed{\text{Ex}}$ Cast a fair dice ($S_0 = \{1, \cdots, 6\}$: equally likely). Consider the events

$$A = \{1, 2, 3\},$$
$$B = \{4, 5\},$$
$$C = \{6\},$$

and the RV

$$X = \begin{cases} 1 & \text{if } A \text{ occurs,} \\ 2 & \text{if } B \text{ occurs} \\ 3 & \text{if } C \text{ occurs,} \end{cases}$$

representing the payoff. If this gamble is repeated sufficiently many times, then one expects to win

$$\begin{cases} 1 \text{ about } \frac{3}{6} \text{ of the times,} \\ 2 \text{ about } \frac{2}{6} \text{ of the times,} \\ 3 \text{ about } \frac{1}{6} \text{ of the times.} \end{cases}$$

$$\Rightarrow 1 \cdot \frac{3}{6} + 2 \cdot \frac{2}{6} + 3 \cdot \frac{1}{6} = \frac{5}{3}$$

"on the average".

$\boxed{\text{DEF}}$ If $\begin{cases} X : \text{RV of discrete type,} \\ S_X : \text{space of } X \\ p_X : \text{PMF of } X \end{cases}$ then

$$E(X) := \sum_{x \in S_X} x \, p_X(x)$$

is called the mathematical expectation or expected value of X, provided the sum exists.

$\boxed{\text{Ex}}$ (Expected value of function of RV)

Flip a fair coin twice, and

$$X = [\#\text{ of heads}].$$
$$Y = u(X), \quad \text{where } u(x) = |x-1|.$$

Then $S_X = \{0, 1, 2\}$ and the PMF p_X of X is

x	0	1	2
$p_X(x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

Now,

(1) $S_Y = \{0, 1\}$.

(2) $p_Y(0) = P(Y=0) = P(X=1) = p_X(1) = \frac{1}{2}$,

$p_Y(1) = P(Y=1) = P(X=0 \text{ or } X=2)$
$= P(X=0) + P(X=2) = p_X(0) + p_X(2) = \frac{1}{2}$.

So,

$$E(Y) = \sum_{y \in S_Y} y \cdot p_Y(y) = 0 \cdot p_Y(0) + 1 \cdot p_Y(1) = \frac{1}{2}.$$

On the other hand, we may as well compute:

$$\begin{aligned} E(Y) &= 0 \cdot p_Y(0) + 1 \cdot p_Y(1) \\ &= 0 \cdot p_X(1) + 1 \cdot p_X(0) + 1 \cdot p_X(2) \\ &= u(0) \cdot p_X(0) + u(1) \cdot p_X(1) + u(2) \cdot p_X(2) \\ &= \sum_{x \in S_X} u(x) \cdot p_X(x). \end{aligned}$$

□

This computation generalizes to:

$\boxed{\text{THM}}$ (LOTUS; Law of the unconscious statistician)

$$E[u(X)] = \sum_{x \in S_X} u(x) \cdot p_X(x).$$

(In the textbook, LOTUS is used to DEFINE the mathematical expectation, perhaps to make proofs look easier.)

Rmks)

- In other words, $$\underbrace{\sum_{y\in S_{u(X)}}}_{\text{space of the RV } u(X)} y\cdot \overbrace{p_{u(X)}}^{\text{PMF of } u(X)}(y) = \sum_{x\in S_X} u(x)\cdot p_X(x)$$

E(u(X)) computed by definition

E(u(X)) computed by LOTUS.

- E(X) need not always exists and is finite.

$\boxed{\text{THM}}$ When exists, $E(\cdot)$ satisfies the following properties:

(a) If c is a constant, then $E(c) = c$.

(b) If c is a constant & X is a RV, then $E(cX) = cE(X)$.

(c) If X_1, X_2 are RVs, then $E(X_1+X_2) = E(X_1)+E(X_2)$.

Pf) We fix an arbitrary RV X throughout.

(a) Regard $c = u(X)$ where $u(x) = c$ is the const. ftn.
Then by LOTUS,

$$E(c) = \sum_{x\in S_X} c\cdot p_X(x) = c\cdot \overbrace{\sum_{x\in S_X} p_X(x)}^{=1} = c.$$

(b) By LOTUS,

$$E(cX) = \sum_{x\in S_X} cx\cdot p_X(x) = c\cdot \sum_{x\in S_X} x\cdot p_X(x) = c\cdot E(X).$$

(c) We only proof the case where $X_1 = u_1(X)$ & $X_2 = u_2(X)$ for some RV X. Then by LOTUS,

$$\begin{aligned} E(X_1+X_2) &= E(u_1(X)+u_2(X)) \\ &= \sum_{x\in S_X} (u_1(x)+u_2(x))\, p_X(x) \\ &= \sum_{x\in S_X} u_1(x)p_X(x) + \sum_{x\in S_X} u_2(x)p_X(x) = E(X_1)+E(X_2) \end{aligned}$$

$\square$

Rmk) • Properties (b) & (c) altogether generalizes to:

(c') $$E\left(\sum_{i=1}^{k} c_i X_i\right) = \sum_{i=1}^{k} c_i E(X_i)$$

for constants c_i's & RVs X_i's. (C') is often called linearity of $E(\cdot)$.

• In general, $E(\cdot)$ do not necessarily commute with other functions.

$\boxed{\text{Ex}}$ For each b,

$\because E(b^2) = b^2.$

$$E[(X-b)^2] = E[X^2 - 2bX + b^2]$$
$$\overset{\text{linearity}}{=} E[X^2] - 2b\,E[X] + b^2.$$

Regarding this as a function of b, it is a quadratic polynomial in b:

$$= (b - E[X])^2 + E[X^2] - (E[X])^2,$$

and this shows that $E[(X-b)^2]$ is minimized at $b = E[X]$. □

$\boxed{\text{Ex}}$ (Geometric distribution) Assume

- A random expr. has $\begin{cases} \text{prob } p \in (0,1) \text{ of SUCCESS} \\ \text{prob } q = 1-p \text{ of FAILURE.} \end{cases}$
- Repeat this expr. independently.
- Set $X = [\#\text{ of trials until the } 1^{st} \text{ success}]$.

(1) Clearly $S_X = \{1, 2, 3, \cdots\}$.

(2) For each $x \in S_X$,

$$\begin{aligned} p_X(x) &= P(X=x) \\ &= P(x-1 \text{ failures then } 1 \text{ success}) \\ &= \underbrace{q \cdot q \cdots q}_{(x-1)\ q\text{'s}} \cdot p = q^{x-1} \cdot p. \end{aligned}$$

Sanity check:

$$\begin{aligned} \sum_{x \in S_X} p_X(x) &= \sum_{x=1}^{\infty} q^{x-1} p = p(1+q+q^2+\cdots) \\ &= \frac{p}{\underbrace{1-q}_{=p}} = 1. \end{aligned}$$

$\Rightarrow$ p_X is indeed a PMF.

Then

$$\begin{aligned} E(X) &= \sum_{x \in S_X} x \cdot p_X(x) \\ &= \sum_{x=1}^{\infty} x \cdot q^{x-1} \cdot p. \end{aligned}$$

(Noting that

$$\begin{aligned} \sum_{n=1}^{\infty} n z^{n-1} &= \sum_{n=1}^{\infty} \frac{d}{dz} z^n = \frac{d}{dz}\left(\sum_{n=1}^{\infty} z^n\right) \\ &= \frac{d}{dz}\left(\frac{z}{1-z}\right) = \frac{1}{(1-z)^2}, \end{aligned}$$

)

we get

$$E(X) = p \cdot \frac{1}{(1-q)^2} = \frac{1}{p}.$$

Similarly,

($$\begin{aligned} \sum_{n=1}^{\infty} n^2 z^{n-1} &= \sum_{n=1}^{\infty} \frac{d}{dz} n z^n = \frac{d}{dz} \sum_{n=1}^{\infty} n z^n \\ &= \frac{d}{dz}\left(\frac{z}{(1-z)^2}\right) = \frac{1+z}{(1-z)^3} \end{aligned}$$)

and hence

$$E(X^2) = \sum_{x \in S_X} x^2 \cdot p_X(x) = \sum_{x=1}^{\infty} x^2 q^{x-1} \cdot p$$

$$= p \cdot \frac{1+q}{(1-q)^3} = \frac{2}{p^2} - \frac{1}{p}. \qquad \square$$