Section 2.1. Random Variables of the Discrete Type

- Often a sample space is either difficult to work with or not a primary object to look at.

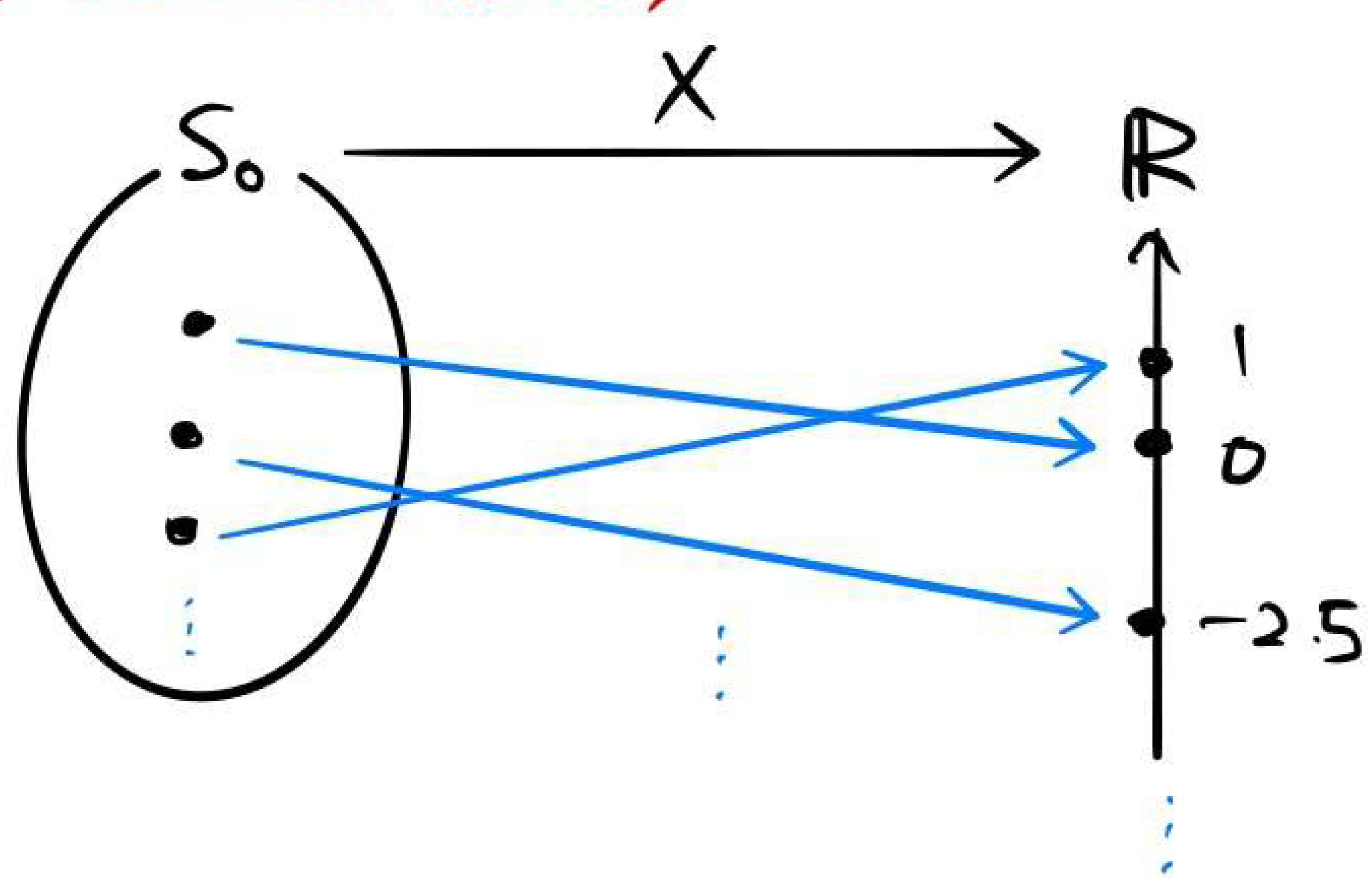
  Ex - Coin flip : $S = \{H, T\}$, outcomes are not numeric.

  - weather forecast : $S = \{$all possible config. of global weather$\}$, almost inaccessible, and may be too large if one is only interested in, say, temp. at LA.

  $\Rightarrow$ A more natural way of modelling is to use "random variable".

$\boxed{\text{DEF}}$ Given a random expr. & sample space S_0, a function $X : S_0 \to \mathbb{R}$ is called a random variable (RV).

($\mathbb{R}$ = set of all real numbers)



$\boxed{\text{Ex}}$

- $S_0 = \{H, T\}$, X specified by $\begin{cases} X(H) = 1 \\ X(T) = 0. \end{cases}$
- $S_0 = \{HH, HT, TH, TT\}$, $X(s) = [\#\text{ of } H\text{'s in } s]$,

  i.e., $X(HH) = 2$, $X(HT) = X(TH) = 1$,
  $X(TT) = 0$.

- $S_0 = \{1, 2, 3, 4, 5, 6\}$, $X(s) = s$.

$\boxed{\text{Ex}}$
- $S_0 = \{(i, j) : i, j = 1, \cdots, 6\}$: outcomes of 2 dice rolls.
- $X(i, j) = i + j =$ [sum of dice]
- X induces events of the form

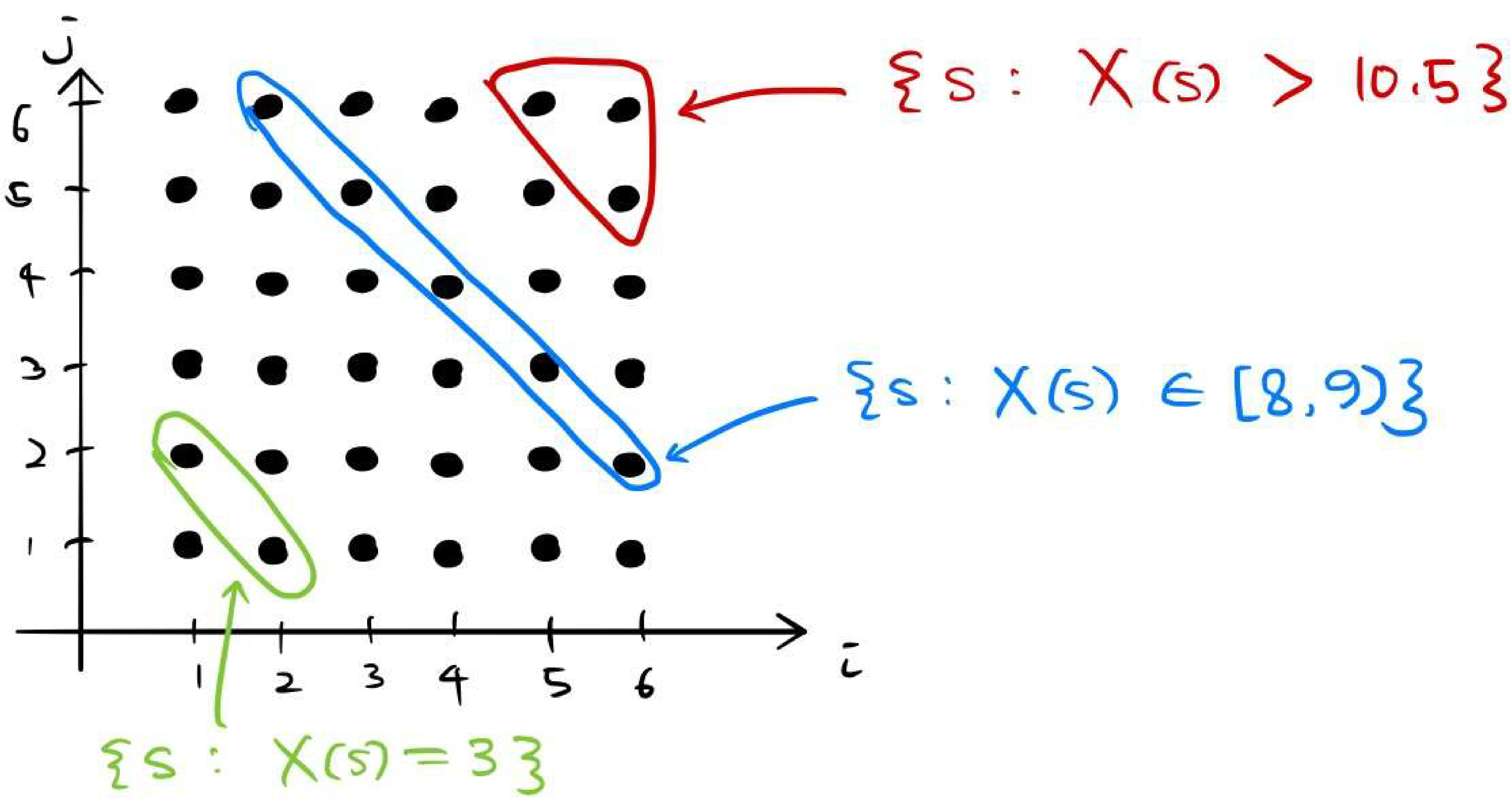
$$\{s : X(s) = x\},$$

or more generally of the form

$$\{s : X(s) \text{ satisfies } \cdots\}$$

for some true/false condition '$\cdots$'.

- Ex



$\boxed{\text{Convention}}$ We abbreviate:

$$P(\{s : X(s) \text{ satisfies } \cdots\}) = P(X \text{ satisfies } \cdots).$$

Ex $P(1 \leq X \leq 3) = P(\{s : 1 \leq X(s) \leq 3\})$,
$P(X \text{ is odd}) = P(\{s : X(s) \text{ is odd}\})$, etc.

- We focus on a special type of RVs:

- For a RV X and $x \in \mathbb{R}$, we write
$$p(x) := P(X = x).$$

$\boxed{\text{DEF}}$ The function p above is called the probability mass function (PMF) of X if there is a set S containing countably many real numbers s.t.

(a) $p(x) > 0$ whenever $x \in S$,

(b) $\sum_{x \in S} p(x) = 1$,

(c) $P(X \in A) = \sum_{x \in A} p(x)$.

In such case,

- X is called a RV of discrete type (or simply discrete RV).
- S is called the space of X (or the support of X).

Rmk) $p(x) = 0$ if $x \notin S$.

$\boxed{\text{DEF}}$ The function
$$F(x) = P(X \leq x)$$
is called the cumulative distribution function (CDF) of X.

Rmk) CDF contains all the probabilistic info. of X only, hence determines the "distribution" of X.

- (Uniform distribution) If S has m elements and

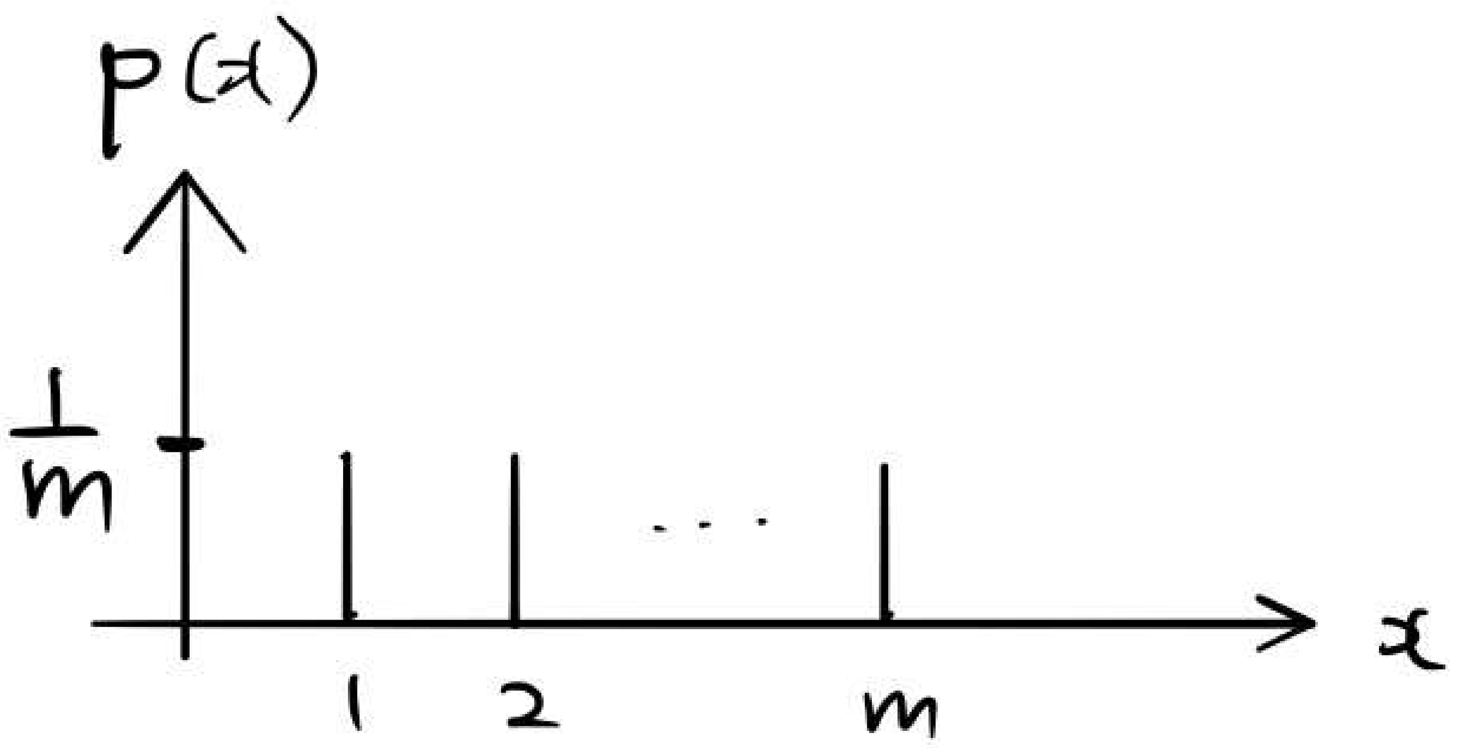
$$p(x) = \frac{1}{m} \quad \text{for } x \in S,$$

then the dist. of X is said to be uniform.

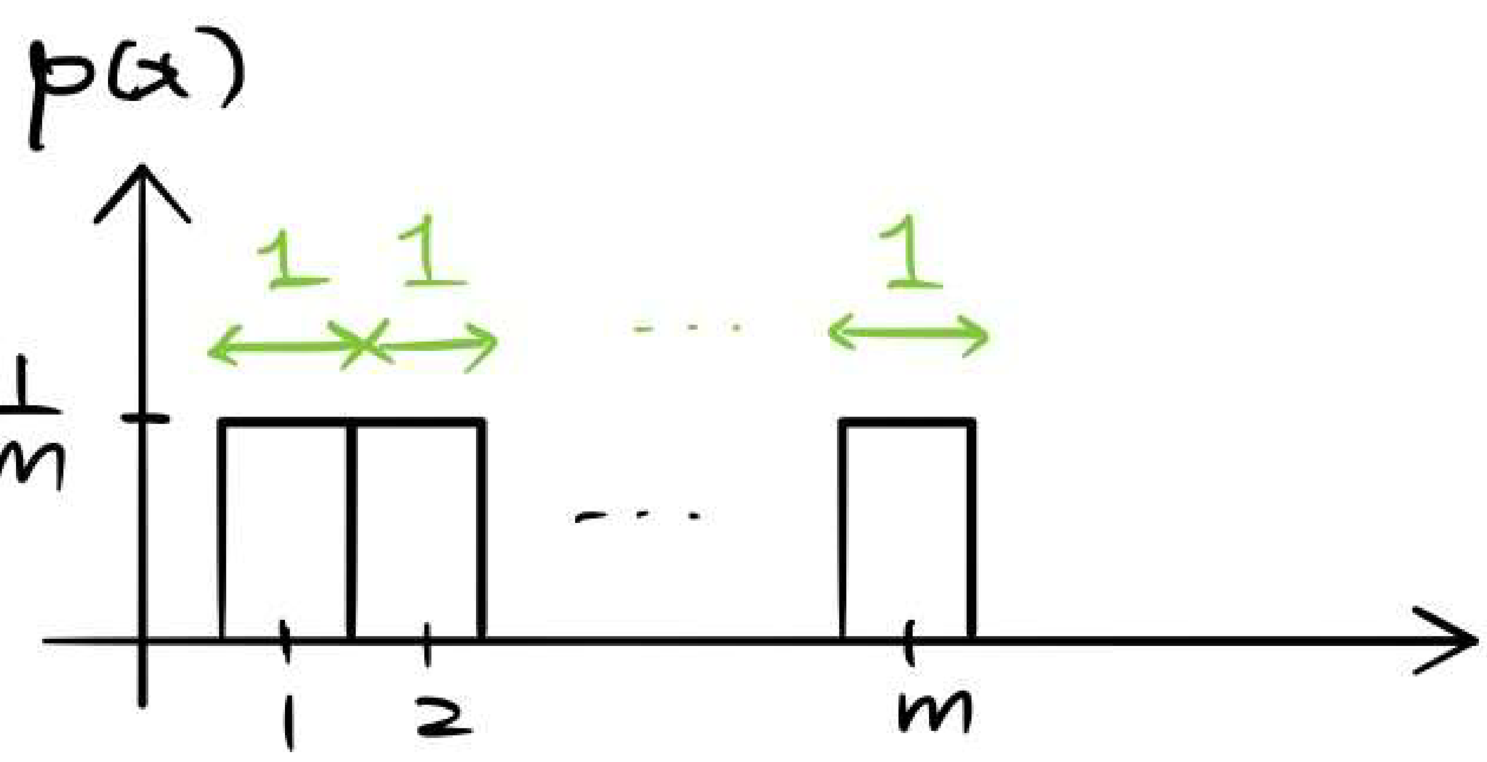
<u>Ex</u> If $S = \{1, \cdots, m\}$, then

$$p(x) = \begin{cases} \frac{1}{m} & \text{for } x = 1, \cdots, m, \\ 0 & \text{otherwise.} \end{cases}$$

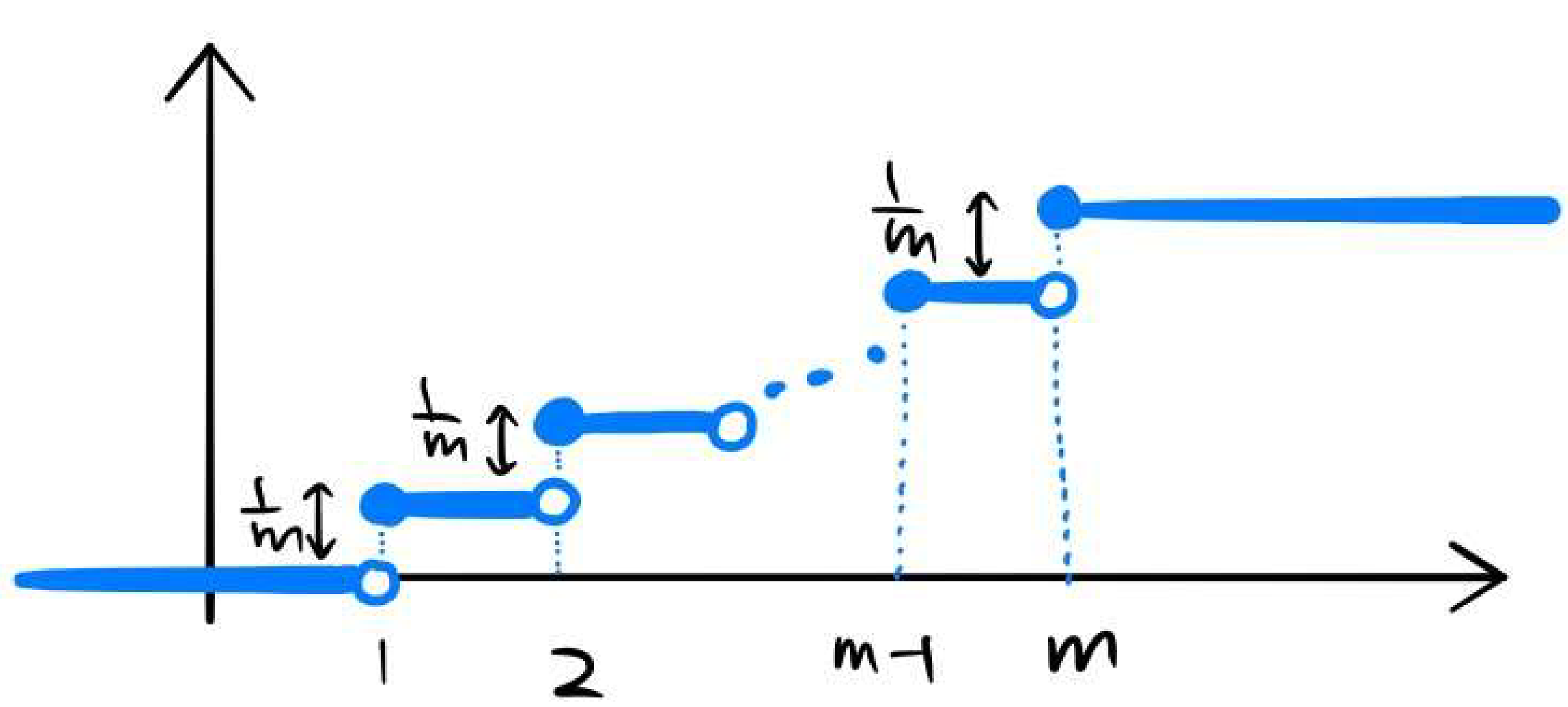
$$F(x) = \begin{cases} 0 & \text{if } x < 1, \\ k/m & \text{if } k \leq x < k+1, \\ 1 & \text{if } k \geq m \end{cases}$$



line graph of p



histogram of p



graph of F

$\boxed{\text{Ex}}$ Roll a fair 4-sided dice twice.

$$S_0 = \{(d_1, d_2) : d_1, d_2 = 1, \cdots, 4\}.$$

Let

$$X = [\text{max. of two outcomes}],$$

i.e.,

$$X(d_1, d_2) = \max\{d_1, d_2\}.$$

Then

- X is discrete RV with $S = \{1, 2, 3, 4\}$
- $\{X=1\} = \{s : X(s) = 1\} = \{(1,1)\}$,

  $\{X=2\} = \{s : X(s) = 2\} = \{(1,2), (2,1), (2,2)\}$

  and so forth. So

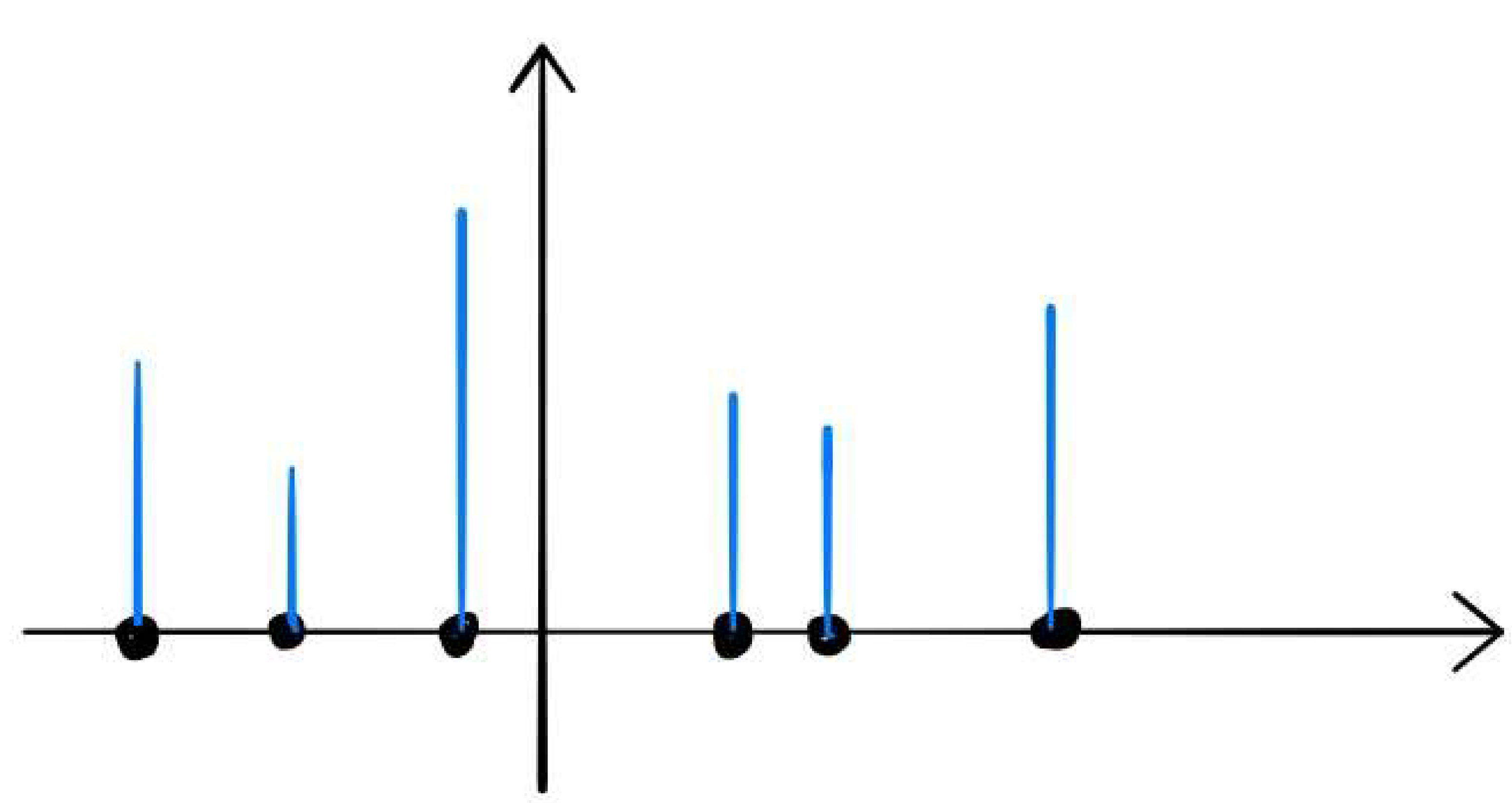
$$p(1) = P(X=1) = \frac{1}{16}$$

$$p(2) = P(X=2) = \frac{3}{16},$$

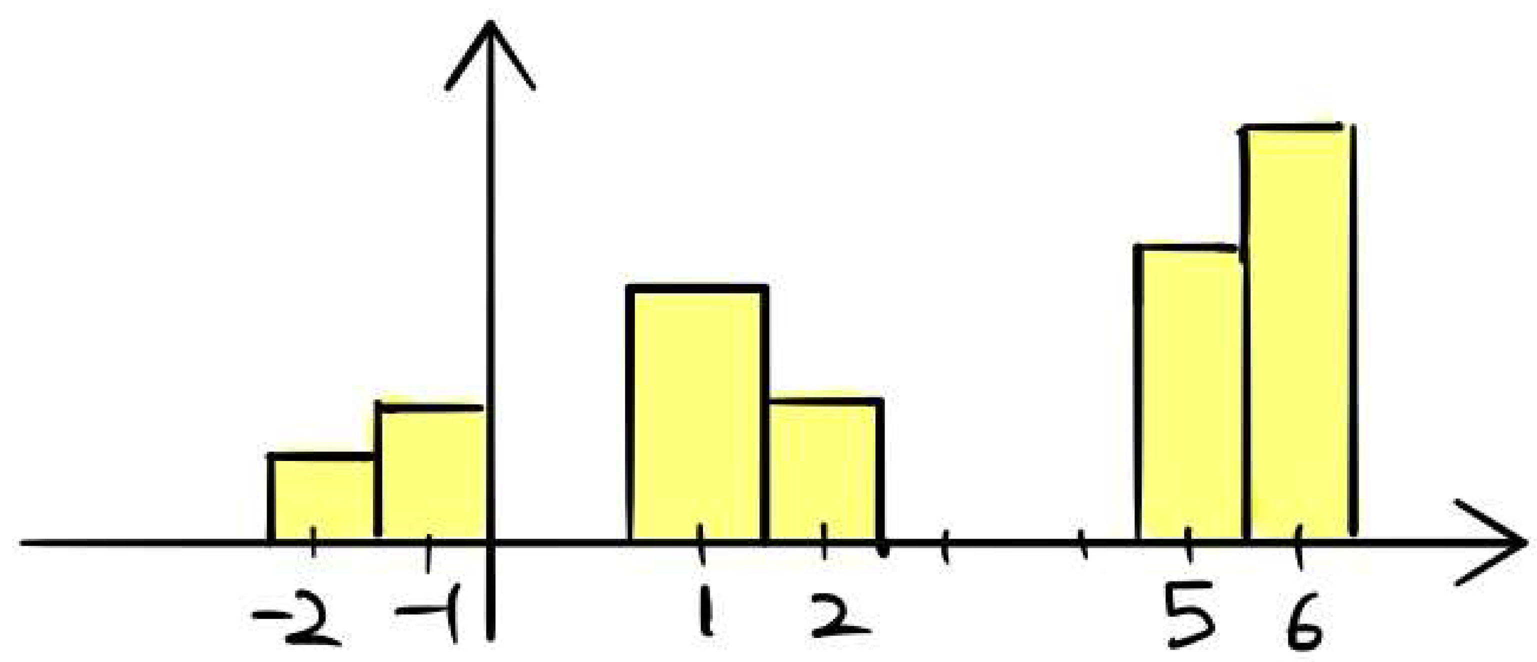
$$\vdots$$

$$p(x) = \begin{cases} \frac{2x-1}{16} & \text{if } x = 1, 2, 3, 4 \\ 0 & \text{o/w.} \end{cases}$$

- (2 visual appreciation)
  - line graph : represent $p(x)$ by vertical line segments joining $(x, 0)$ to $(x, p(x))$ at each $x \in S$.



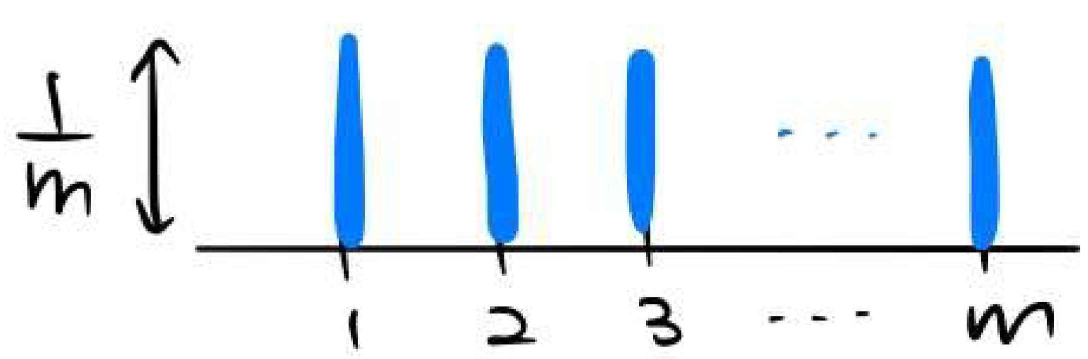
- probability histogram : If X assumes only integer values, represent $p(x)$ by a rectangle of height $p(x)$ and the base of length 1 centered at x for each $x \in S$.



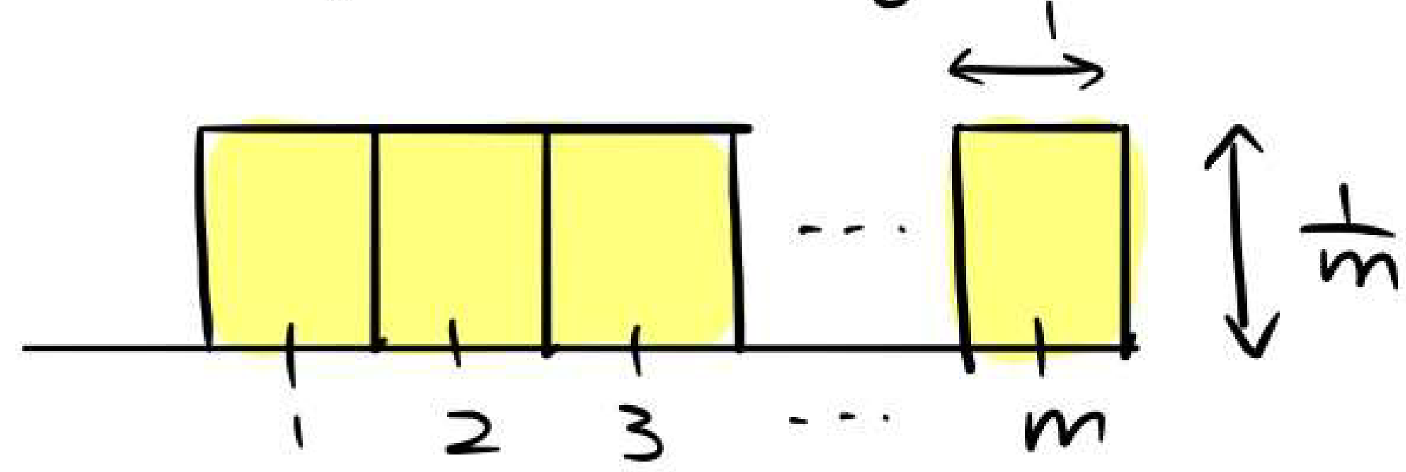
- relative frequency histogram : prob. histogram applied to relative freq. in place of pmf.

$\boxed{\text{EX}}$ For the uniform distribution on $S = \{1, \cdots, m\}$,

line graph:



prob. histogram:

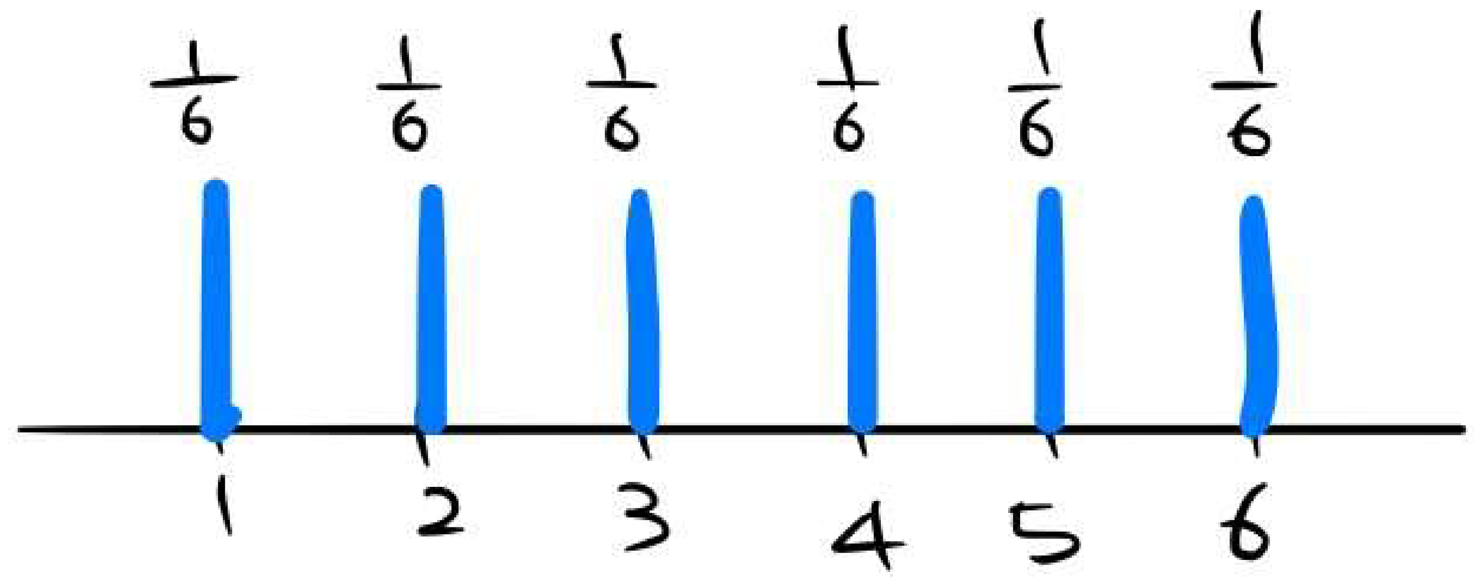


<u>Rmk*)</u> Different RVs can have the same distribution.

$\boxed{\text{Ex}}$ Roll a fair dice twice. $S_0 = \{(i,j) : i,j = 1, \cdots, 6\}$.

Let $\begin{cases} X_1 = [1^{st} \text{ roll}], & \text{i.e., } X(i,j) = i \\ X_2 = [2^{nd} \text{ roll}], & \text{i.e., } X(i,j) = j. \end{cases}$

Then X_1 and X_2 are different RVs but they have the same distribution, namely



the uniform distribution over $\{1, \cdots, 6\}$. Saying differently,

$$p_{X_1} = [\text{PMF of } X_1] = [\text{PMF of } X_2] = p_{X_2}.$$