Section 1.4. Independent Events.

DEF

- Events A and B are **independent** if
$$P(A\cap B) = P(A)P(B).$$
Otherwise, they are dependent.

- Events $A_1, \cdots, A_k$ are **pairwise independent** if
$$P(A_i\cap A_j) = P(A_i)P(A_j) \quad \text{for any } i\neq j,$$
i.e., each pair A_i, A_j $(i\neq j)$ is independent.
- Events $A_1, \cdots, A_k$ are **(mutually) independent** if
$$P(A_{i_1}\cap\cdots\cap A_{i_m}) = P(A_{i_1})\cdots P(A_{i_m})$$
for any m and $i_1<\cdots<i_k$.

Ex Events A, B, C are

(1) pairwise indep. $\Longleftrightarrow$
$$P(A\cap B) = P(A)P(B),$$
$$P(B\cap C) = P(B)P(C),$$
$$P(C\cap A) = P(C)P(A).$$

(2) mutually indep. $\Longleftrightarrow$ —//— and
$$P(A\cap B\cap C) = P(A)P(B)P(C).$$

Ex Flip a fair coin twice. We may set
$$S = \{HH, HT, TH, TT\},$$
and all outcomes equally likely. Then
$$A := \{\text{Hs on the } 1^{st} \text{ flip}\} = \{HH, HT\}$$
$$B := \{\text{Ts on the } 2^{nd} \text{ flip}\} = \{HT, TT\}.$$
$$C := \{\text{Ts on both flips}\} = \{TT\}.$$
Then
$$P(B|A) = \frac{P(B\cap A)}{P(A)} = \frac{1/4}{2/4} = \frac{1}{2} = P(B).$$

I.e., knowing whether A occurred or not does not affect the prob. of B. But

$$P(C|A) = \frac{P(C\cap A)}{P(A)} = 0 \neq \frac{1}{4} = P(C),$$

so ———— // ———— affects the prob. of C. $\square$

Q How to check A & B are indep?

(1) Check if $P(A\cap B) = P(A)P(B)$ holds, or

(2) Check if $P(A|B) = P(A)$ or $P(B) = P(B|A)$ holds.

$\boxed{\text{Ex}}$ ▶ red dice & white dice are rolled. Let

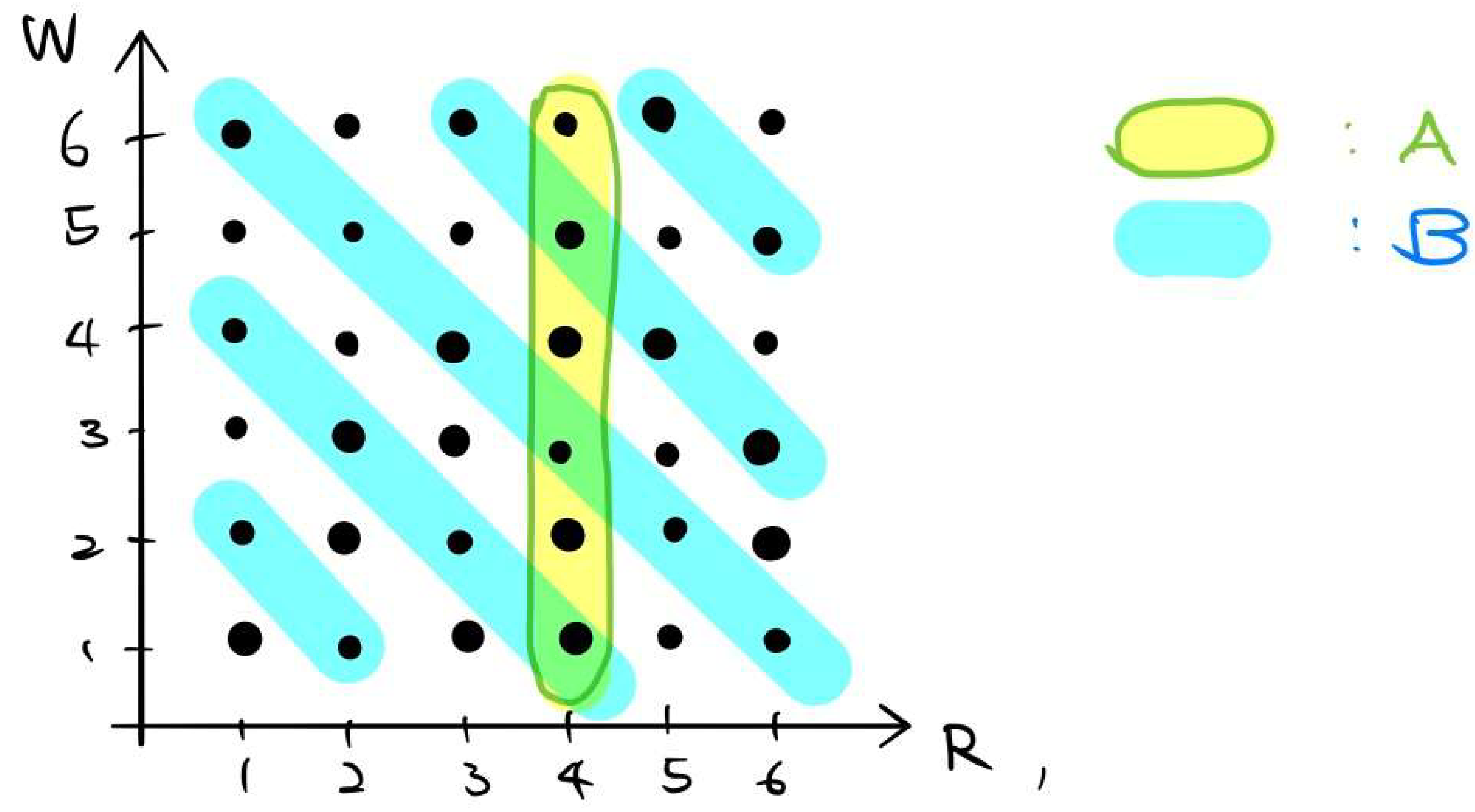
$A = \{$ 4 on red dice $\}$

$B = \{$ sum of dice is odd $\}$.

Then

$$S = \{(1,1), (1,2), \cdots, (1,6), (2,1), \cdots, (6,6)\},$$

and



so

$$P(A)P(B) = \frac{6}{36}\cdot\frac{18}{36} = \frac{3}{36} = P(A\cap B).$$

▶ Now set

$$C = \{\text{ sum of dice is } 7\}$$
$$D = \{\text{ sum of dice is } 11\}.$$

Then

$$P(A)P(C) = \frac{6}{36}\cdot\frac{6}{36} = \frac{1}{36} = P(A\cap C)$$
$$P(A)P(D) = \frac{6}{\cancel{36}}\cdot\frac{2}{36} = \frac{1}{108} \neq 0 = P(A\cap D).$$

So

A & C : indep, but A & D : dep. $\square$

$\boxed{\text{THM 1.4-1}}$ If A & B : indep., then

(a) A & B' : indep.

(b) A' & B : indep.

(c) A' & B' : indep.

Pf) A & B : indep. $\overset{\text{by def.}}{\Longleftrightarrow}$ $P(A\cap B) = P(A)P(B)$. Then

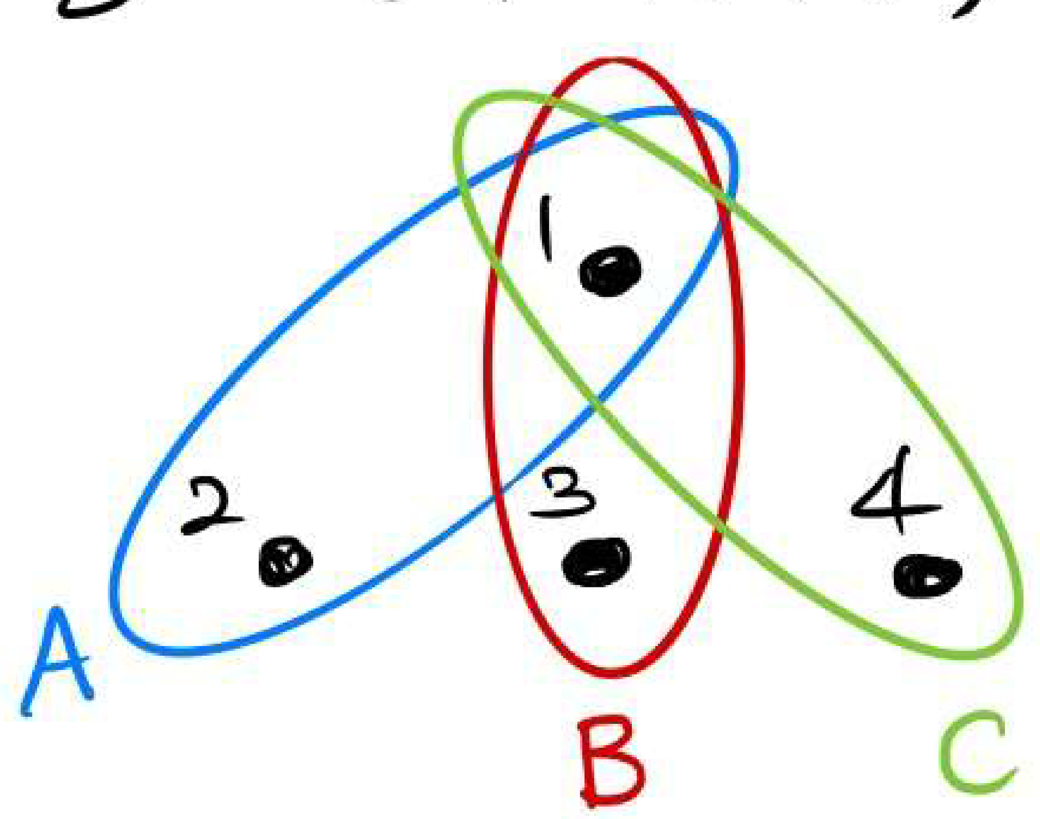
$$\begin{aligned} P(A) &= P(A\cap B) + P(A\cap B') \\ &= P(A)P(B) + P(A\cap B') \end{aligned}$$

$$\begin{aligned} \Rightarrow P(A\cap B') &= P(A) - P(A)P(B) \\ &= P(A)(1-P(B)) \\ &= P(A)P(B'). \end{aligned}$$

$\Rightarrow$ A & B' indep. $\square$

$\boxed{\text{Ex}}$ (1) mutually indep. $\Longrightarrow$ pairwise indep. always holds.

(2) Consider: $S = \{1, 2, 3, 4\}$ equally likely, and set



Then

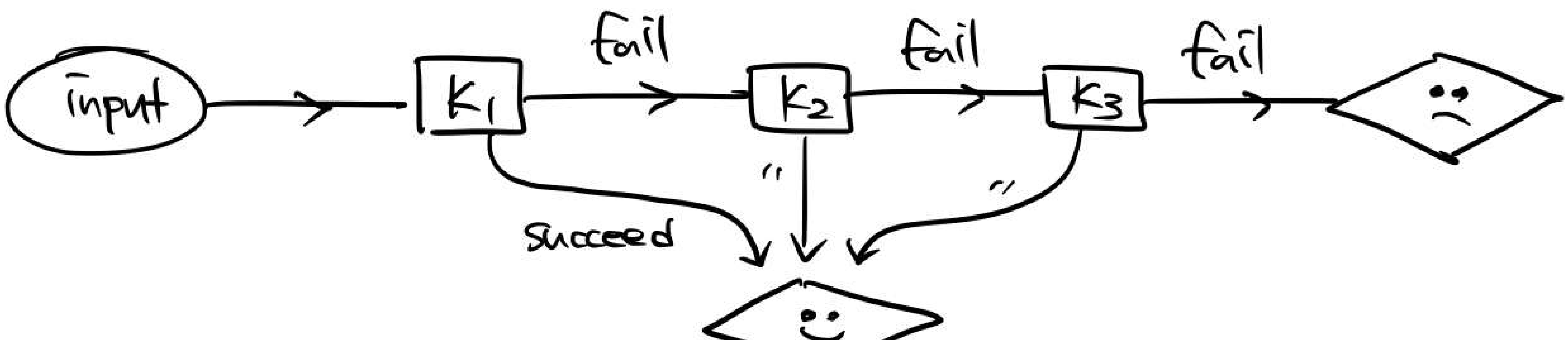
$$P(A \cap B) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A)P(B)$$

and likewise for other pairs, so A, B, C : pairwise indep. But

$$P(A \cap B \cap C) = \frac{1}{4} \neq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = P(A)P(B)P(C),$$

so A, B, C : NOT mutually indep. $\square$

Ex

- 

- Let $A_i = \{\text{system } K_i \text{ fails}\}$, $P(A_i) = 0.85$
- Assume A_i's are indep. Then

$$\begin{aligned} P(☺) &= 1 - P(☹) \\ &= 1 - P(A_1 \cap A_2 \cap A_3) \\ &= 1 - P(A_1)P(A_2)P(A_3) \\ &= 1 - (0.85)^3. \end{aligned}$$

$\square$

Ex If A, B, C : mutually indep, then

(a) A & $B \cup C$: indep

(b) A & $B \cap C$: indep

(c) A & $B \setminus C$: indep.

<u>Pf)</u> (b) : $P(A \cap (B \cap C)) = P(A)P(B)P(C) = P(A)P(B \cap C)$.

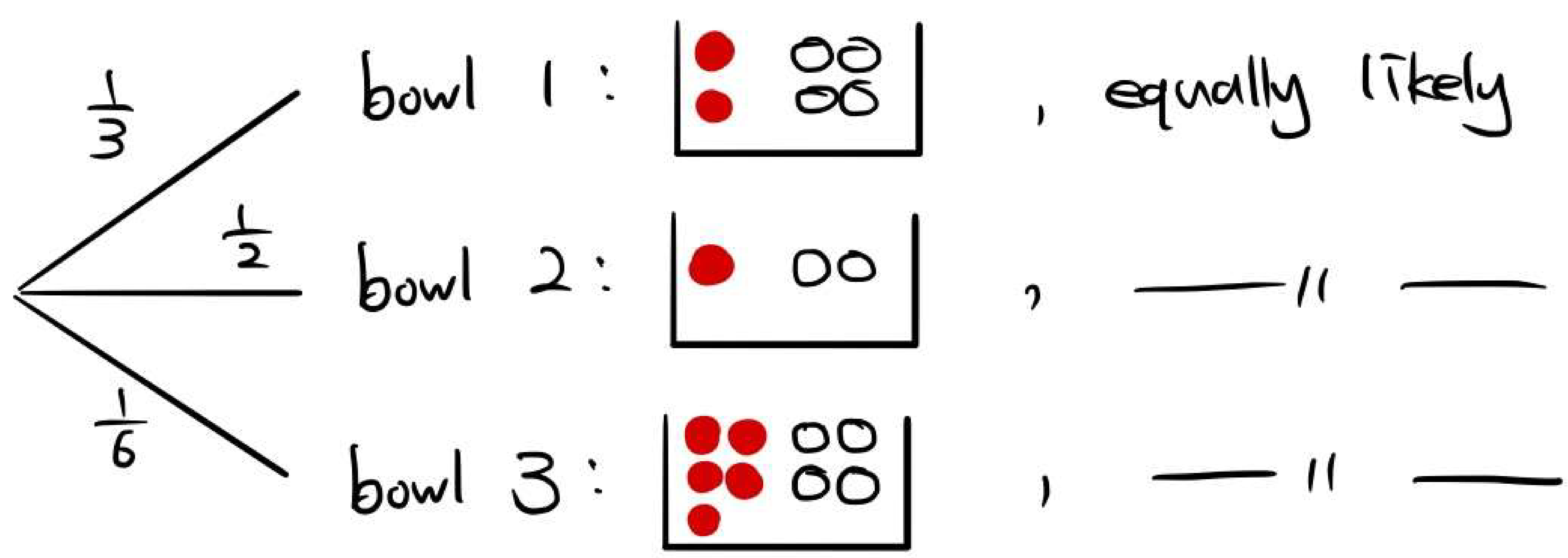
(a) : Suffices to consider the case $P(A) > 0$. (Why?) Then

$$\begin{aligned} P(B \cup C \mid A) &= P(B \mid A) + P(C \mid A) - P(B \cap C \mid A) \\ &\overset{(a)}{=} P(B) + P(C) - P(B \cap C) \\ &= P(B \cup C). \end{aligned}$$

$\square$

Section 1.5. Bayes' Theorem

$\boxed{\text{Ex}}$



$$\begin{cases} B_i = \{ i^{th} \text{ bowl chosen} \} \\ R = \{ \text{red chip drawn} \}. \end{cases}$$

We know: $P(B_i)$'s and $P(R|B_i)$'s.

Q1 $P(R) = ?$

$$P(R) = P(B_1)P(R|B_1) + P(B_2)P(R|B_2) + P(B_3)P(R|B_3)$$
$$= \frac{1}{3}\cdot\frac{2}{6} + \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{6}\cdot\frac{5}{9} = \frac{4}{9}.$$

Q2 $P(B_1|R)$ = [prob. that 1^{st} bowl chosen, knowing red chip drawn]

$= ?$

$$P(B_1|R) = \frac{P(B_1 \cap R)}{P(R)}$$
$$= \frac{P(B_1)P(R|B_1)}{P(B_1)P(R|B_1) + P(B_2)P(R|B_2) + P(B_3)P(R|B_3)}$$
$$= \frac{2}{8}.$$

Similar computations show: $P(B_2|R) = \frac{1}{8}$, $P(B_3|R) = \frac{5}{8}$.

Rmk) Even though $P(B_3)$ is small, $P(B_3|R)$ is large.

<u>Setting</u>
- $B_1, \cdots, B_m$: partition of S
  (= mutually exclusive & exhaustive)
- prior probabilities $P(B_i)$'s are positive.

<u>Then</u>

(1) (Law of total probability)

$$P(A) = \sum_{i=1}^{m} P(A \cap B_i) = \sum_{i=1}^{m} P(B_i)P(A|B_i).$$

(2) (Bayes' theorem) The posterior probabilities are given by

$$P(B_k|A) = \frac{P(B_k)P(A|B_k)}{\sum_{i=1}^{m} P(B_i)P(A|B_i)}.$$

$\boxed{\text{Ex}}$ Pap smear : cervical cancer test. Let

$T^- = \{\text{test is neg.}\}$ $\quad$ $C^- = \{\text{not cancer}\}$

$T^+ = \{\text{test is pos.}\}$ $\quad$ $C^+ = \{\text{cancer}\}$.

Assume we know

- Prob. of false neg. $= P(T^-|C^+) = 0.16$
- Prob. of false pos. $= P(T^+|C^-) = 0.10$.
- $P(C^+) = 0.00008$.

Then

$$P(C^+|T^+) = \frac{P(C^+ \cap T^+)}{P(T^+)}$$

$= 1 - P(T^-|C^+) = 0.84$

$$= \frac{P(C^+)P(T^+|C^+)}{P(C^+)P(T^+|C^+) + P(C^-)P(T^+|C^-)}$$

$$= \frac{0.00008 \cdot 0.84}{0.00008 \cdot 0.84 + 0.99992 \cdot 0.10}$$

$= 0.000672\cdots$. $\quad$ very small! ☹