Note 03

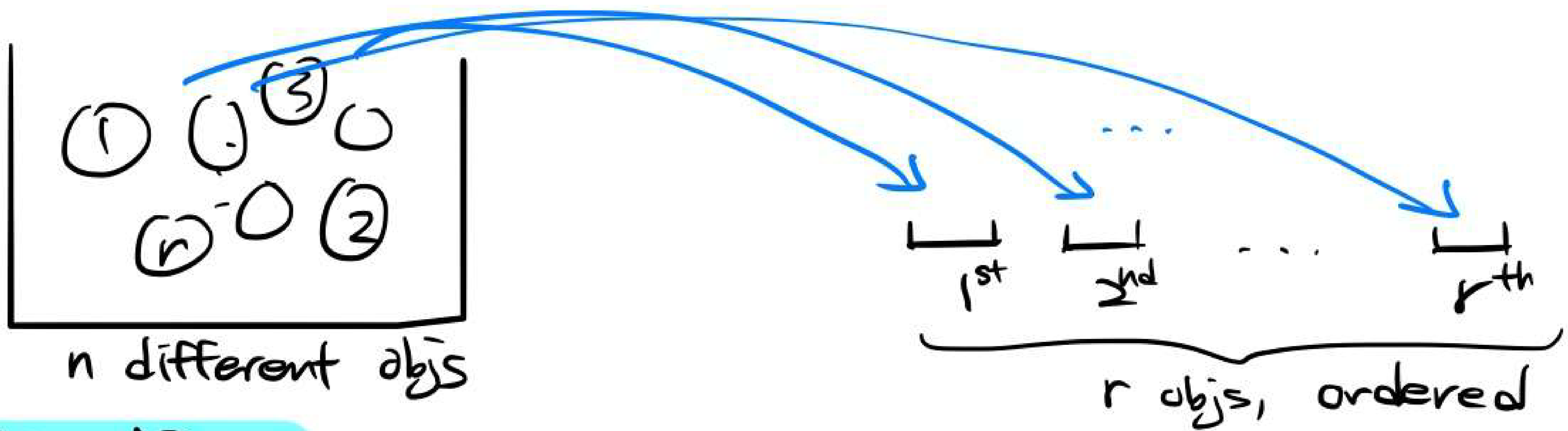
Section 1.2

Recall) 2 major schemes of sampling without replacement:

① **Permutation**

$$_nP_r = \frac{n!}{(n-r)!}$$

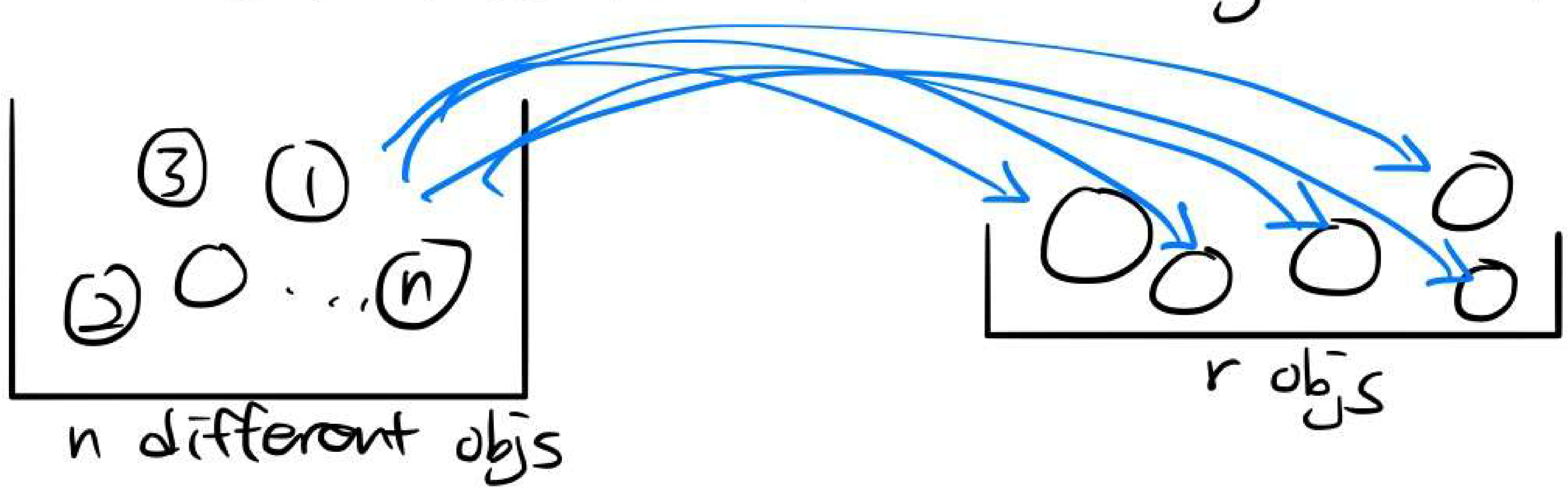
$= [\#$ of permutations of n objs taken r at a time$]$



② **Combination**

$$_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$= [\#$ of combinations of n objs taken r at a time$]$



$\boxed{\text{Ex}}$ [# of five-card hands drawn from 52 cards]

$$= \binom{52}{5} = \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = 2{,}598{,}960.$$

Assume all possible five-card hands are equally likely, consider:

$A = \{$ 5 spades $\}$,

$B = \{$ 3 kings & 2 queens $\}$.

Then

(1) $$N(A) = \underbrace{\binom{13}{5}}_{\substack{\text{5 out of 13} \\ \text{spades}}} \times \underbrace{\binom{39}{0}}_{\substack{\text{0 out of 39} \\ \text{nonspades}}} = 1287.$$

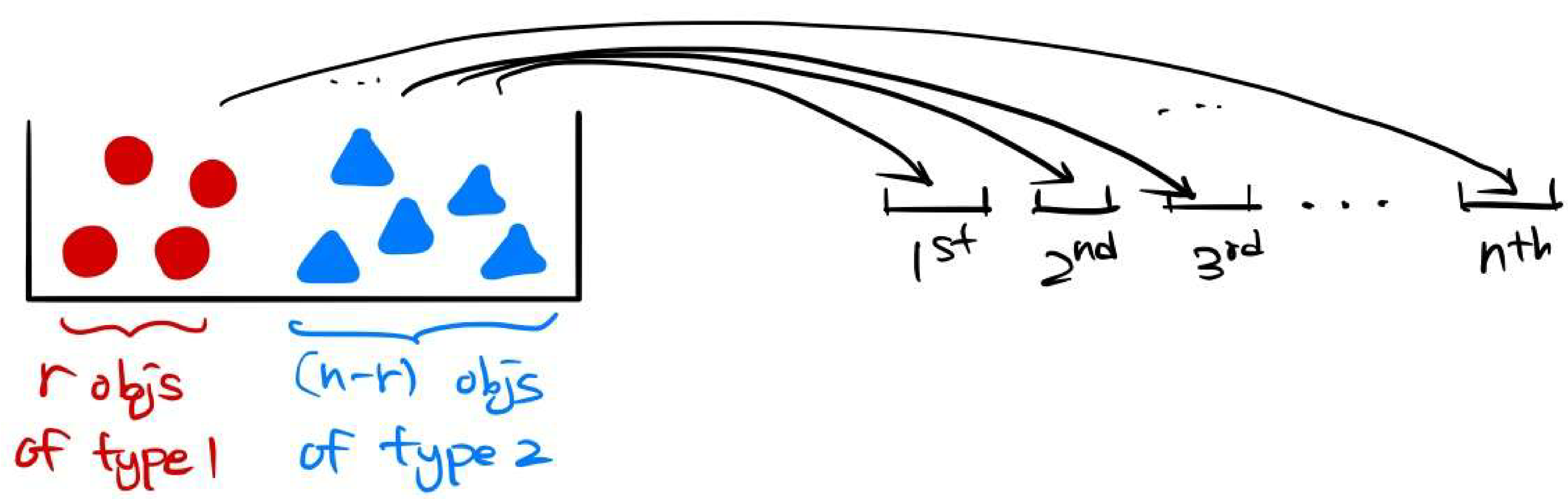
$$\Rightarrow P(A) = \frac{N(A)}{N(S)} = \frac{1287}{2958960} \approx 0.000495\cdots.$$

(2) $$N(B) = \underbrace{\binom{4}{3}}_{\substack{\text{3 out of 4} \\ \text{kings}}} \times \underbrace{\binom{4}{2}}_{\substack{\text{2 out of 4} \\ \text{kings}}} \times \underbrace{\binom{44}{0}}_{\substack{\text{0 out of 44} \\ \text{non kings \& nonqueens}}} = 24.$$

$$\Rightarrow P(B) = \frac{N(B)}{N(S)} = \frac{24}{2958960} \approx 0.0000092\cdots.$$ □

③ Distinguishable Permutations

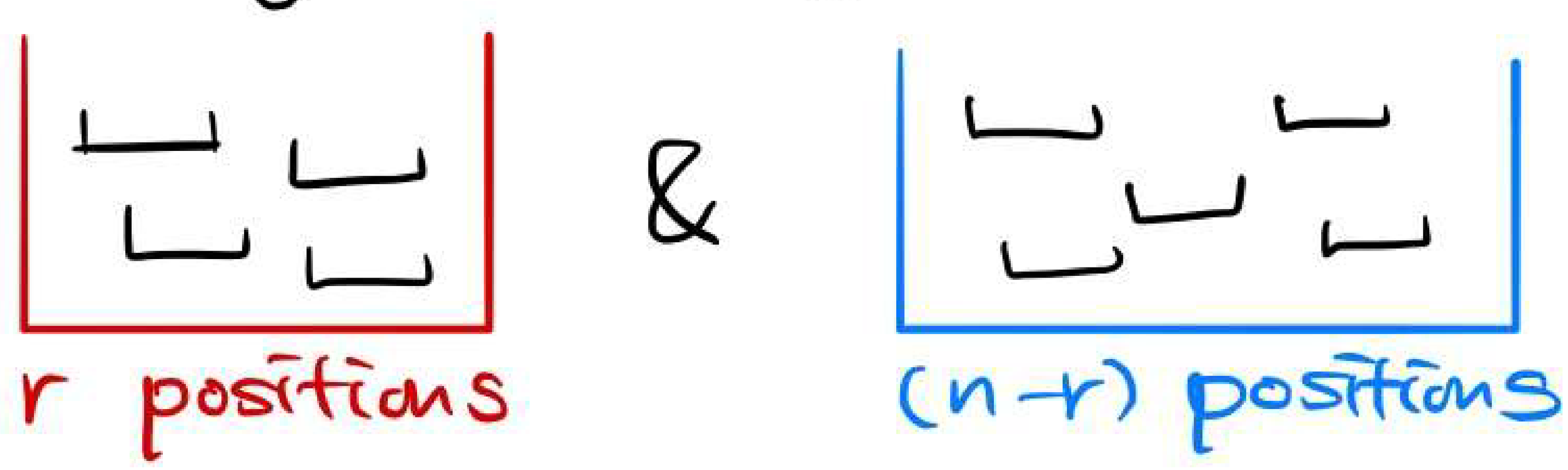
Q Given $\left\{ \begin{array}{l} r \text{ objs of one type} \\ n-r \text{ objs of another type} \end{array} \right\}$, how many ways to arrange them?



$\boxed{\text{DEF}}$ Any such arrangement is called a distinguishable permutation.

- [# of distinguishable permutations]

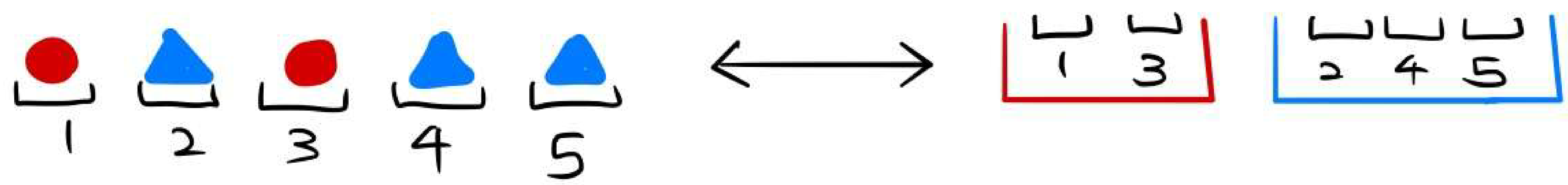
$=$ [# of ways of assignments :



]

$$= \binom{n}{r}\underbrace{\binom{n-r}{n-r}}_{=1} = \binom{n}{r}.$$

$\boxed{\text{Ex}}$



$\boxed{\text{Ex}}$ (Binomial Theorem)

$$(a+b)^1 = a+b,$$
$$(a+b)^2 = aa+ab+ba+bb,$$
$$(a+b)^3 = aaa+aab+aba+abb + baa+bab+bba+bbb,$$
$$\vdots$$

$\Longrightarrow$ The expansion of $(a+b)^n$ consists of 2^n possible arrangements of distinguishable perms of size n & types a/b.

$\Rightarrow$ [coef. of $a^r b^{n-r}$ in the expansion of $(a+b)^n$]

$= [\#$ of ways of arranging r a's & $(n-r)$ b's$]$

$= \binom{n}{r}$.

$$\Rightarrow \quad (a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}. \qquad \square$$

- Similar argument applies to 3 or more types: Let $n = n_1 + \cdots + n_s$.

$[\#$ of ways of arranging n_i objs of type i, $i = 1, \cdots, s]$

$$= \binom{n}{n_1}\binom{n-n_1}{n_2} \cdots \binom{n-n_1-\cdots-n_{s-1}}{n_s}$$

$$= \frac{n!}{n_1!\,\cancel{(n-n_1)!}} \cdot \frac{\cancel{(n-n_1)!}}{n_2!\,\cancel{(n-n_1-n_2)!}} \cdots \frac{\cancel{(n-n_1-\cdots-n_{s-1})!}}{n_s!\,\cancel{0!}}$$

$$= \frac{n!}{n_1! \cdots n_s!}$$

$\boxed{\text{DEF}}$ $\binom{n}{n_1, \cdots, n_s} := \dfrac{n!}{n_1! \cdots n_s!}$ is called a multinomial coefficient.

Section 1.3 Conditional Probability

Q What is the natural way of defining Probability set function on the "new" sample space $B \subseteq S$?

DEF The conditional probability of A given B is

$$P(A|B) := \frac{P(A\cap B)}{P(B)}$$

provided $P(B) > 0$.

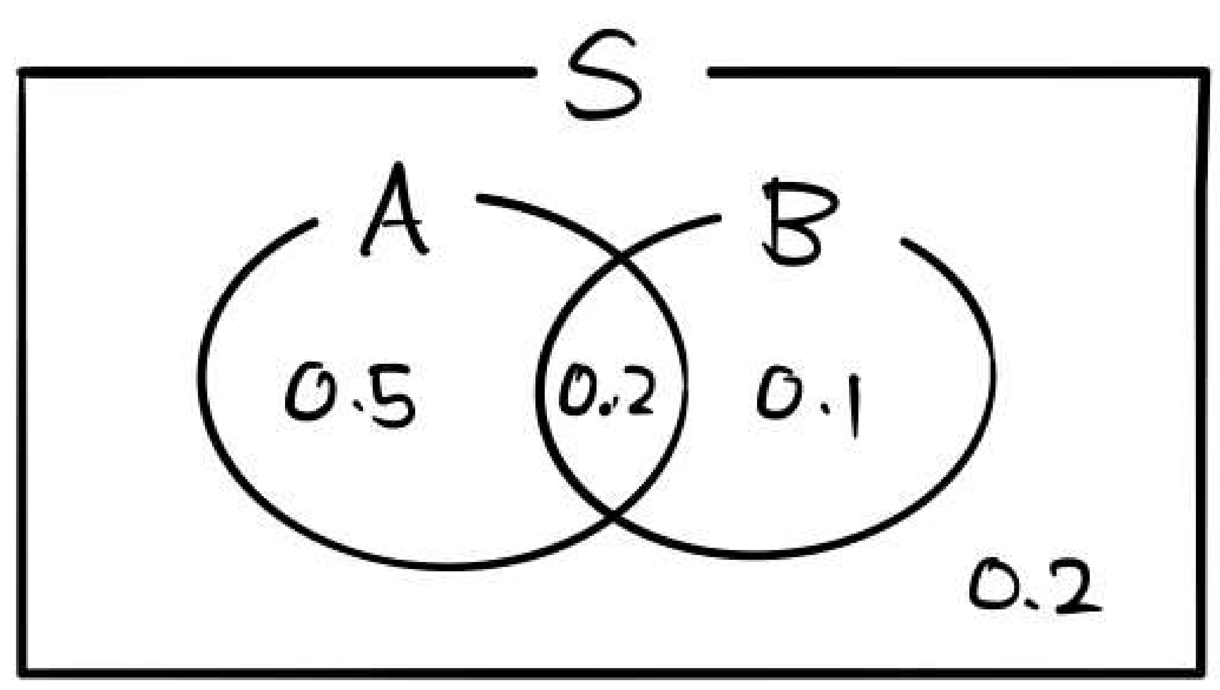
Ex If all outcomes are equally likely,

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{N(A\cap B)/\cancel{N(S)}}{N(B)/\cancel{N(S)}} = \frac{N(A\cap B)}{N(B)}. \quad \square$$

Ex Suppose $P(A) = 0.7$, $P(B) = 0.3$, $P(A\cap B) = 0.2$.

$$\Rightarrow \quad P(A|B) = \frac{P(A\cap B)}{P(B)}$$

$$= \frac{0.2}{0.3} = \frac{3}{2}.$$



$\square$

PROP $P(\cdot|B)$ is indeed a Prob. set-ftn on B:

(a) $P(A|B) \geq 0$.

(b) $P(B|B) = P(B\cap B)/P(B) = 1$.

(c) A_k's mutually exclusive

$$\Rightarrow P(A_1\cup A_2 \cup \cdots |B) = \frac{P((A_1\cup A_2\cup\cdots)\cap B)}{P(B)}$$

$$= \frac{P((A_1\cap B)\cup(A_2\cap B)\cup\cdots)}{P(B)}$$

$$= \frac{P(A_1\cap B) + P(A_2\cap B) + \cdots}{P(B)}$$

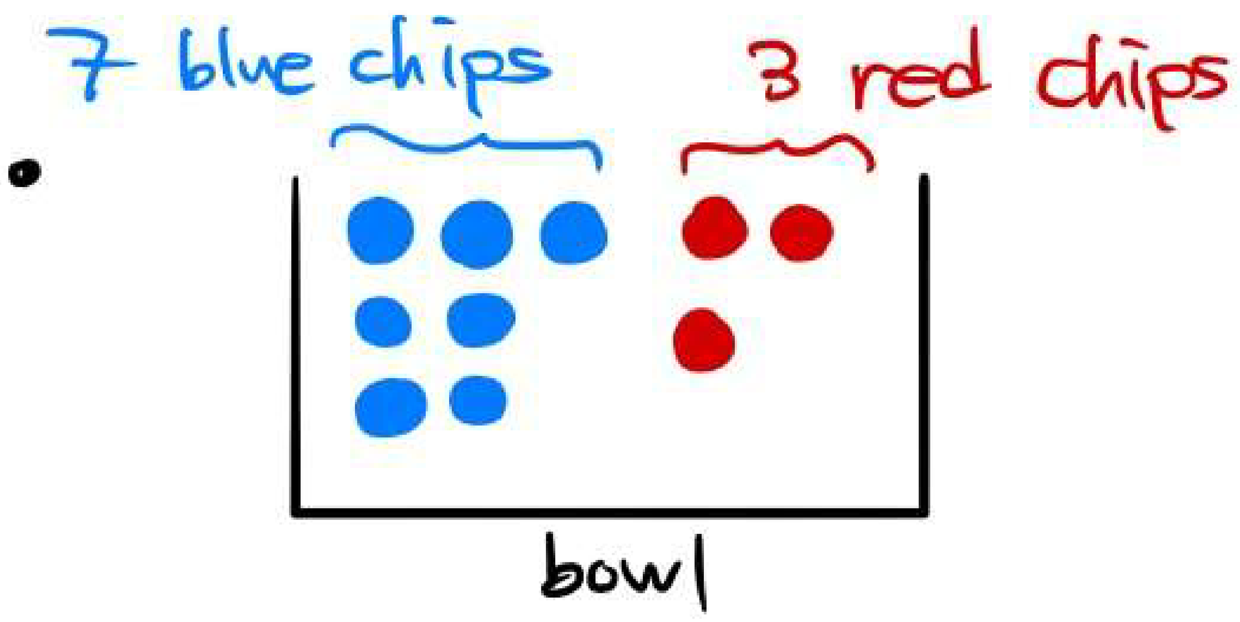
$$= P(A_1|B) + P(A_2|B) + \cdots.$$

__Rmk)__ All theorems in Section 1.1 apply to $P(\cdot \mid B)$!!

$\boxed{\text{PROP}}$ (Multiplication rule)

$$P(A \cap B) = P(A)\,P(B \mid A) \quad \text{provided } P(A) > 0$$
$$= P(B)\,P(A \mid B) \quad \text{provided } P(B) > 0.$$

$\boxed{\text{Ex}}$ •



- Draw 2 chips w/o replacement,
- $\begin{cases} A = \{ 1^{st} \text{ draw is red} \} \\ B = \{ 2^{nd} \text{ draw is blue} \} \end{cases}$

What is $P(A \cap B) = ?$

- Reasonable to assign: $P(A) = \frac{3}{10}$, $P(B \mid A) = \frac{7}{9}$.

$$\Rightarrow P(A \cap B) = P(A)P(B \mid A) = \frac{3}{10} \cdot \frac{7}{9} = \frac{7}{30}.$$

- Alternatively, let $S = \{\text{set of all possible draws}\}$, then

$$\begin{cases} N(S) = {}_{10}P_2 = 90, \\ N(A \cap B) = \binom{3}{1}\binom{7}{1} = 21. \end{cases}$$

$$\Rightarrow P(A \cap B) = \frac{N(A \cap B)}{N(S)} = \frac{21}{90} = \frac{7}{30}. \qquad \square$$

__Rmk)__ Multi. rule extends to several events:

$$P(A \cap B \cap C) = P(A \cap B)\,P(C \mid A \cap B)$$
$$= P(A)\,P(B \mid A)\,P(C \mid A \cap B),$$

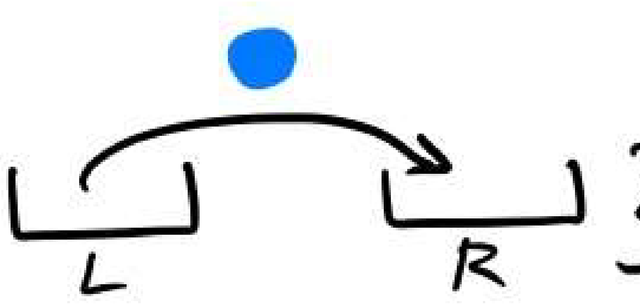
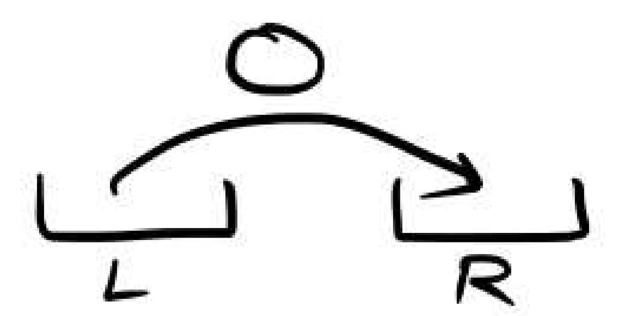
provided $P(A \cap B) > 0$.

$\boxed{\text{Ex}}$

left pocket right pocket

- Transfer one marble from left $\to$ right, then draw a marble from the right.
- $\underline{Q}$ The prob. that a blue marble is drawn?

$\underline{\text{Sol)}}$ Write

BL $= \{$  $\}$, WL $= \{$  $\}$,

BR $= \{$  $\}$.

Then

$$\begin{aligned} P(BR) &= P(BL \cap BR) + P(WL \cap BR) \\ &= P(BL)\,P(BR \mid BL) + P(WL)\,P(BR \mid WL) \\ &= \frac{5}{9} \cdot \frac{5}{10} + \frac{4}{9} \cdot \frac{4}{10} = \frac{41}{90}. \end{aligned}$$

$\boxed{\text{Ex}}$ Assume A_1, A_2, A_3 : mutually exclusive &

$$\begin{aligned} P(A_1) &= 0.3 & P(B \mid A_1) &= 0.6 \\ P(A_2) &= 0.2 & P(B \mid A_2) &= 0.7 \\ P(A_3) &= 0.2 & P(B \mid A_3) &= 0.8. \end{aligned}$$

What is $P(B \mid A_1 \cup A_2 \cup A_3)$?

$\underline{\text{Sol)}}$

$$\begin{aligned} P(B \mid A_1 \cup A_2 \cup A_3) &= \frac{P(B \cap A_1) + P(B \cap A_2) + P(B \cap A_3)}{P(A_1) + P(A_2) + P(A_3)} \\ &= \frac{0.3 \times 0.6 + 0.2 \times 0.7 + 0.2 \times 0.8}{0.3 + 0.2 + 0.2} \approx 0.686. \end{aligned}$$