OH: W 3:30–5
F 1–2:30

HW 3, 5, 10 in Section 1.1
3, 7, 10, 17 in Section 1.2

Lecture 2

1.2. Method of Enumerations

GOAL Develop counting technique useful in determining # of outcomes of random experiments.

- **Multiplication Principle** Suppose
  - Experiment 1 (E_1) has n_1 possible outcomes
  - Experiment 2 (E_2) has n_2 possible outcomes, for each possible outcome of E_2.

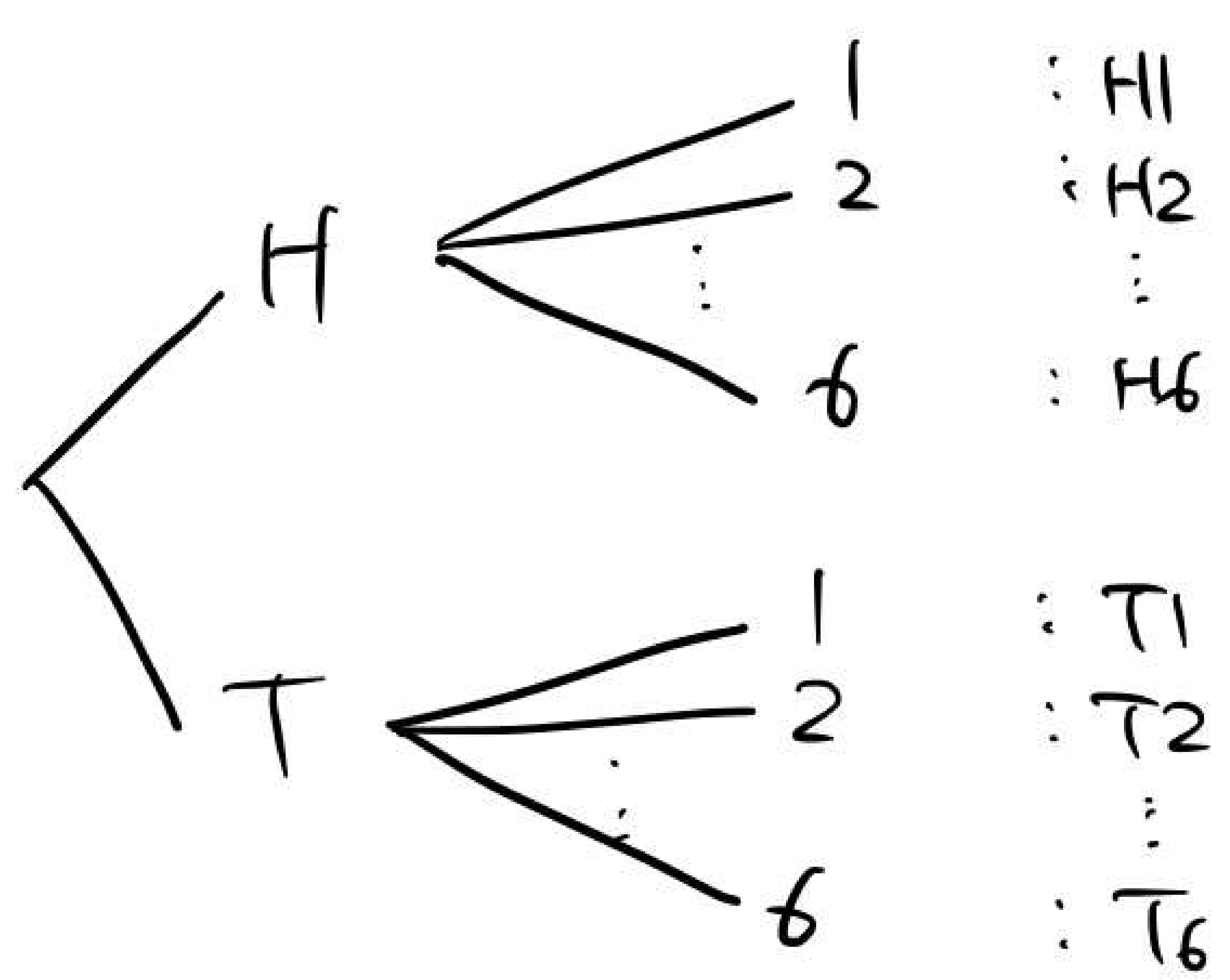
Then the expr. E_1E_2 (performing E_1 and then E_2) has n_1n_2 possible outcomes.

$\boxed{\text{EX}}$ $\begin{cases} E_1 : \text{coin flip } (H, T) \\ E_2 : \text{dice roll } (1, \cdots, 6) \end{cases}$

Then E_1E_2 has $2 \times 6 = 12$ possible outcomes

H1 H2 ⋯ H6
T1 T2 ⋯ T6.

This may be visualized using a tree diagram:



$\boxed{\text{Ex}}$ How many different varieties of pizzas are possible if

- size : S / M / L
- crust : thin / normal / pan
- topping : 12 types (you may select from 0 to 12)

Sol) There are 14 experiments:

- which size?
- which crust?
- add topping #1? (Y/N)

⋮

- add topping #12? (Y/N).

So by MP,

$$3 \times 3 \times 2^{12}$$

varieties are possible. $\square$

$\boxed{\text{Ex}}$ How many 2 letter code words are possible using the letters in CAT, if

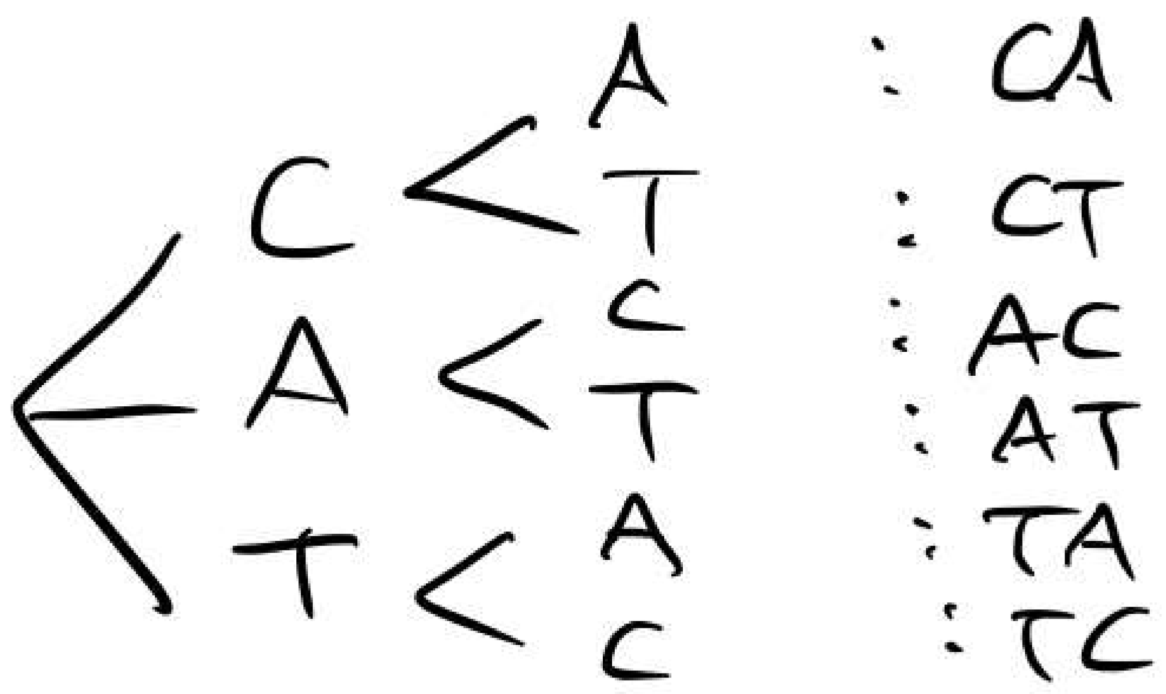
(1) the letters may not be repeated?

(2) the letters may be repeated?

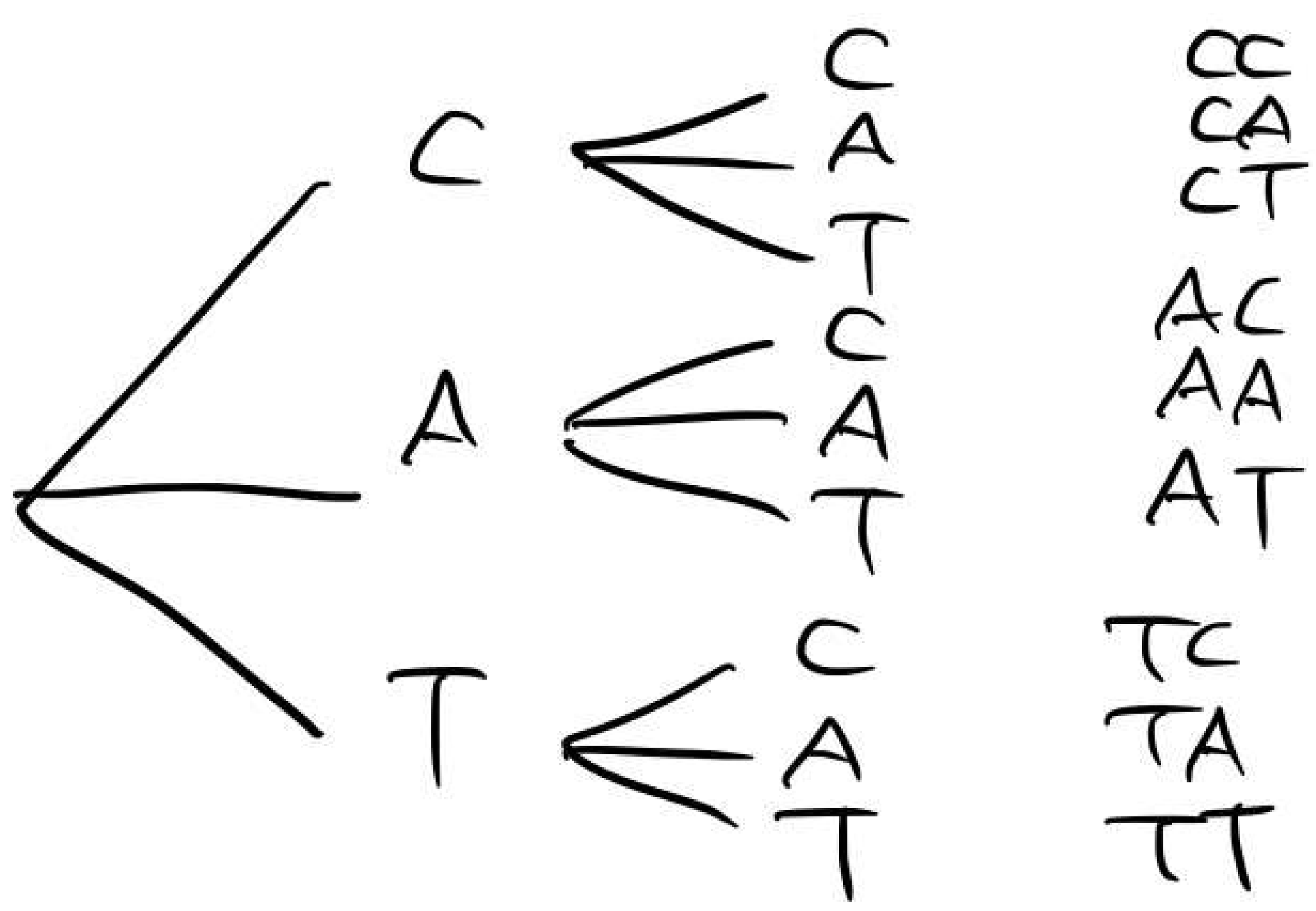
Sol) Imagine filling letters of the code word one by one.

$$\underset{E_1}{\square} \quad \underset{E_2}{\square}$$

(1) In such scenario, $\left\{ \begin{array}{l} E_1 \text{ has } 3 \text{ choices,} \\ E_2 \text{ has } 2 \text{ choices.} \end{array} \right\} \Rightarrow 3 \times 2 = 6.$



(2) In such scenario, both E_1 & E_2 has 3 choices.
$\Rightarrow 3 \times 3 = 3^2 = 9$.



$\blacksquare$

- This example demonstrates two types of sampling schemes. Consider the problem of

  selecting r objects from n objects.

$\boxed{\text{DEF}}$ - **sampling with replacement**: an object is selected and then replaced before the next object is selected.
- **sampling without replacement**: an object is not replaced after it has been selected.

- The order in which objects are selected may or may not be important.

DEF When the order of selection is noted, the selected set is called an ordered sample.

- So we have 4 possible cases:

  - w/o replacement
    - ordered : permutation
    - unordered : combination
  - w/ replacement
    - ordered : (perm. w/ repetition)
    - unordered : (combi. w/ repetition)

① Permutation

- Q How many ways to fill r positions w/ n different objects?

DEF
- Each of arrangements of n different objects is called a permutation of the n objects.
- Each of arrangements of r objects chosen from the set of n different objects is called a permutation of n objects taken r at a time.

DEF
- (factorial) $n! := n \times (n-1) \times \cdots \times 2 \times 1$, $\quad 0! := 1$.
- $_nP_r := \dfrac{n!}{(n-r)!}$

- Then $[\#\text{ of perms of } n \text{ objs}] = n \times (n-1) \times \cdots \times 1 = n!$,

  and

$$[\#\text{ of perms of } n \text{ objs taken } r \text{ at a time}]$$
$$= n \times (n-1) \times \cdots \times (n-r+1)$$
$$= {}_nP_r.$$

② Combination

- <u>Q</u> How many ways to choose r objects, without ordering, from the set of n different objects? (I.e., how many subsets of size r?)

- Let C = [# of subsets of size r chosen from n different objs].

  Then

$$C \cdot (r!) = [\text{\# of subsets of size } r \cdots] \times [\text{\# of ways of arranging } r \text{ objs}] = [\text{\# of ordered samples of size } r \cdots] = {}_nP_r.$$

  So

$$C = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}.$$

old-fashioned notation

$\boxed{\text{DEF}}$ - (binomial coefficient) $\binom{n}{r} := {}_nC_r := \frac{n!}{r!(n-r)!}$

$\boxed{\text{DEF}}$ Each of the (unordered) subsets is called a combination of n objects taken r at a time.

$\boxed{\text{THM}}$ (Binomial Theorem)

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$$

<u>Idea)</u> - Expanding $(a+b)^n$ gives 2^n terms, each of which being a product of n variables from $\{a, b\}$.

- [# of times $a^r b^{n-r}$ appears in the expansion]

$$= \left[\begin{array}{c} \text{filling } r \text{ positions with } a\text{'s, chosen from} \\ \underbrace{\_\overset{\times}{}\_\overset{\times}{}\_\overset{\times}{} \cdots \overset{\times}{}\_}_{n}, \\ \text{and then filling the rest } (n-r) \text{ pos. w/ } b\text{'s} \end{array} \right]$$

$$= \binom{n}{r}.$$

$\square$

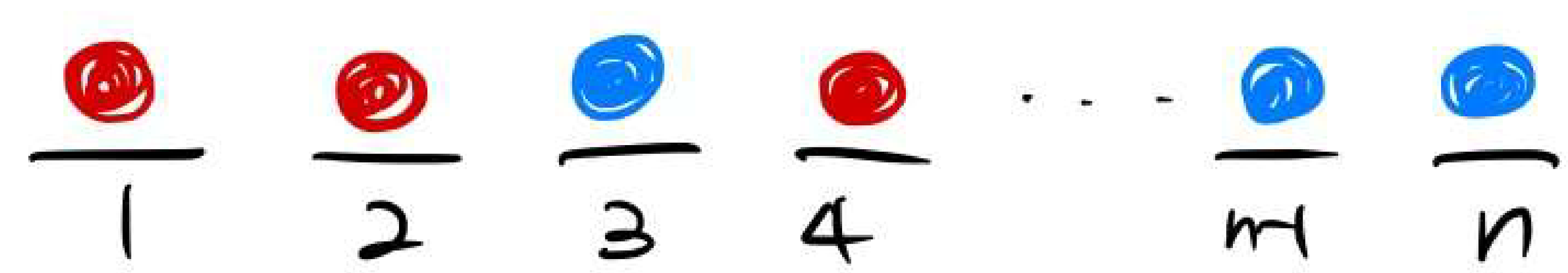
③ Distinguishable Permutation.

- [# of permutations of n object, $\left\{ \begin{array}{l} r \text{ of one type, \&} \\ n-r \text{ of another type} \end{array} \right\}$]

$$= \binom{n}{r}$$

Indeed,

[assigning positions to r red balls (●) & $n-r$ blue balls (●)



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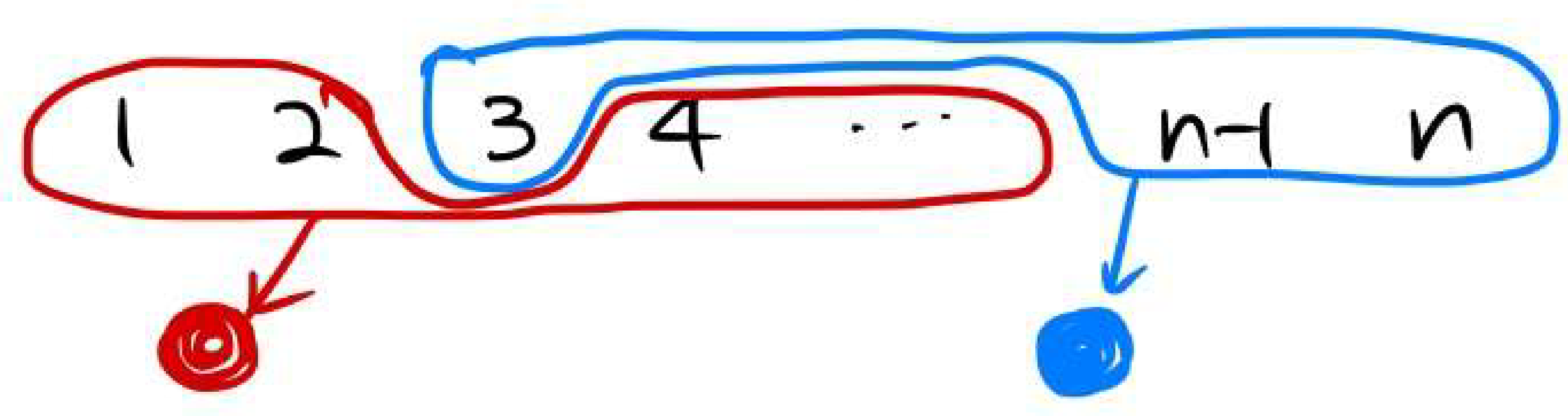
= [choosing r positions for ● out of n pos. & $n-r$ pos. for ● out of the remaining $n-r$ pos.



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$$= \binom{n}{r} \times \binom{n-r}{n-r} = \binom{n}{r}.$$

- Similar argument shows: If $n = n_1 + \cdots + n_s$, then

$$
\begin{aligned}
&[\#\text{ of permutations of } n \text{ objs, } n_i \text{ of type } i \text{ for } i=1,\cdots,s] \\
&= [\#\text{ of ways of choosing } n_1 \text{ pos. for type 1 from } n \text{ pos}] \\
&\times [\text{———— '' ————} \; n_2 \text{ pos. for type 2 from } n-n_1 \text{ pos}] \\
&\vdots \\
&\times [\text{———— '' ————} \; n_s \text{ pos. for type } s \text{ from } n-(n_1+\cdots+n_{s-1}) \text{ pos}] \\
&= \binom{n}{n_1}\times\binom{n-n_1}{n_2}\times\binom{n-n_1-n_2}{n_3}\times\cdots\times\binom{n-n_1-\cdots-n_{s-1}}{n_s} \\
&= \frac{n!}{n_1!\,n_2!\cdots n_s!}
\end{aligned}
$$

$\boxed{\text{DEF}}$ – (multinomial coefficient) $\binom{n}{n_1, n_2, \cdots, n_s} := \frac{n!}{n_1!\,n_2!\cdots n_s!}$,
where $n_1 + \cdots + n_s = n$.

(Obviously, $\binom{n}{r} = \binom{n}{r, n-r}$.)

$\boxed{\text{THM}}$ (Multinomial Theorem)

[coef. of $a_1^{n_1}\cdots a_s^{n_s}$ in the expansion of $(a_1+\cdots+a_s)^n$]

$$= \binom{n}{n_1, n_2, \cdots, n_s}.$$