| Math 170E Lecture 1, Winter 2020 | Homework 9 | Due March 13, in class |
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- Some exercises are taken from R.V. Hogg, E. Tanis, D. Zimmerman, *Probability and Statistical Inference*, 10th Ed., Pearson.
- The questions marked with a star ($\star$) are either more difficult or of the form that is not intended for an exam. Nonetheless, they are worth thinking about.

## Exercises

**1.** Let $X$ have the PDF

$$f_X(x) = \begin{cases} 4x^3, & 0 < x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the PDF of $Y = X^2$.

**2.** The PDF of $X$ is

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

**(a)** Find the CDF of $X$.

**(b)** Describe how an observation of $X$ can be simulated using a uniform distribution over $(0, 1)$.

**3.** Let $X$ have a **logistic distribution** with PDF

$$f_X(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \qquad -\infty < x < \infty.$$

Show that

$$Y = \frac{1}{1+e^{-X}}$$

has a uniform distribution over $(0,1)$.

**4.** Let $X$ have the uniform distribution over $(-1,3)$. Find the PDF of

$$Y = X^2.$$

*Hint: The function $u(x) = x^2$ is **not** one-to-one over $(-1,3)$, and so, the change of variables technique as described in the class does not work directly to this case. However, the CDF technique still applies.*

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**5.** Let $X_1$ and $X_2$ be independent random variables with respective binomial distributions $b(3, \frac{1}{2})$ and $b(5, \frac{1}{2})$.

**(a)** Find $P(X_1 = 2, X_2 = 4)$.

**(b)** Find $P(X_1 + X_2 = 7)$.

**6.** Let $X_1$ and $X_2$ be independent random variables with probability density functions

$$f_{X_1}(x) = \begin{cases} 2x, & 0 < x < 1, \\ 0, & \text{elsewhere,} \end{cases} \qquad f_{X_2}(x) = \begin{cases} 4x^3, & 0 < x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

**(a)** Compute $P(0.5 < X_1 < 1,\ 0.4 < X_2 < 0.8)$.

**(b)** Find $E(X_1^2 X_2^3)$.

**7.** Suppose two independent claims are made on two insured homes, where each claim has PDF

$$f(x) = \begin{cases} \dfrac{4}{x^5}, & 1 < x < \infty, \\ 0, & \text{elsewhere,} \end{cases}$$

in which the unit is \$1000. Find the expected value of the larger claim.

*Hint: If $X_1$ and $X_2$ are independent claims and $Y = \max\{X_1, X_2\}$, then the CDF of $Y$ satisfies*

$$F_Y(y) = P(Y \leq y) = P(X_1 \leq y)P(X_2 \leq y) = F_X(y)^2,$$

*where $X$ is another independent claim. Using this, determine the PDF of $Y$ and then compute $E(Y)$.*

**8.** A device contains three components, each of which has a lifetime in hours with the PDF

$$f(x) = \begin{cases} \dfrac{2x}{10^2} e^{-(x/10)^2}, & 0 < x < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

The device fails with the failure of one of the components. Assuming independent lifetimes, what is the probability that the device fails in the first hour of its operation?

*Hint: If $X_1$, $X_2$, and $X_3$ are three independent lifetimes and $Y = \min\{X_1, X_2, X_3\}$, then*

$$\begin{aligned} P(Y \leq 1) &= 1 - P(Y > 1) \\ &= 1 - P(X_1 > 1)P(X_2 > 1)P(X_3 > 1). \end{aligned}$$

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**9.** Let $X_1, \ldots, X_n$ be independent Poisson random variables with respective means $\lambda_1, \ldots, \lambda_n$. Show that the sum

$$Y = \sum_{i=1}^{n} X_i$$

is Poisson with mean

$$\lambda = \sum_{i=1}^{n} \lambda_i.$$

*Hint: Recall that X is Poisson with mean $\lambda$ if and only if its MGF is given by $M_X(t) = \exp(\lambda(e^t - 1))$.*

**10.** The number $X$ of sick days taken during a year by an employee follows a Poisson distribution with mean 2. Let us observe four such employees. Assuming independence, compute the probability that their total number of sick days exceeds 10.

*Hint: Use Problem 9 to determine the distribution of the total number of sick days.*

**11.** Let $X_1$, $X_2$, $X_3$ denote a random sample of size 3 from a gamma distribution with $\alpha = 7$ and $\theta = 5$.

**(a)** Find the MGF of $Y = X_1 + X_2 + X_3$.

**(b)** How is $Y$ distributed?

*Hint: Recall that X has a gamma distribution with parameters α and θ if and only if its MGF is given by* $M_X(t) = (1 - \theta t)^{-\alpha}$.