Math 170E Lecture 1, Winter 2020	Homework 8	Due March 2, in class
Name:		**UID:**

- Some exercises are taken from R.V. Hogg, E. Tanis, D. Zimmerman, *Probability and Statistical Inference*, 10th Ed., Pearson.

4.3 Conditional Distributions

1. Let X and Y have the joint PMF

$$p(x,y) = \frac{x+y}{32}, \qquad x = 1,2, \quad y = 1,2,3,4.$$

(a) Find the marginal PMFs $f_X(x)$ and $f_Y(y)$.

(b) Find $p_{X|Y}(x|y)$ and $P(X = 2 \mid Y = 3)$.

(c) Find $p_{Y|X}(y|x)$ and $P(1 \leq Y \leq 3 \mid X = 1)$.

Math 170E Lecture 1, Winter 2020	Homework 8	Due March 2, in class
Name:		**UID:**

2. Let W equal the weight of laundry soap in a 1-kilogram box that is distributed in Southeast Asia. Suppose that

$$P(W < 1) = 0.02 \qquad \text{and} \qquad P(W > 1.072) = 0.08.$$

Call a box of soft light, good, or heavy, depending on whether $\{W < 1\}$, $\{1 \leq W \leq 1.072\}$, or $\{W > 1.072\}$, respectively. In $n = 50$ independent observations of these boxes, let

$$X = [\text{number of light boxes}],$$
$$Y = [\text{number of good boxes}]$$

(a) What is the joint PMF of X and Y?

(b) Given that $X = 3$, how is Y distributed conditionally?

(c) Determine $E(Y \mid X = 3)$.

3. Suppose that the conditional expectation of Y, given $X = x$, is linear in x, i.e.,

$$E(Y \mid X = x) = a + bx$$

for some constants a and b. The aim of this exercise is to determine the values of a and b.

(a) Compute $\mu_Y = E(Y)$ using the law of total expectation and deduce that a and b satisfy $a + b\mu_X = \mu_Y$.

(b) Compute $E(XY)$ using the law of total expectation and deduce that a and b satisfy $a\mu_X + bE(X^2) = E(XY)$.

(c) *(Optional, not graded)* Using the previous parts, determine the values of a and b, then verify that the linearity assumption leads to

$$E(Y \mid X = x) = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X),$$

i.e., $E(Y \mid X = x)$ is the least square regression line.

Math 170E Lecture 1, Winter 2020	Homework 8	Due March 2, in class
Name:		**UID:**

4. A fair six-sided die is rolled 30 independent times. Let X be the number of ones and Y the number of twos.

(a) What is the joint PMF of X and Y?

(b) Find the conditional PMF of X, given $Y = y$.

(c) Compute $E[X^2 - 4XY + 3Y^2]$.

5. Suppose that X has a Poisson distribution with the rate λ and suppose the conditional distribution of Y, given $X = x$, is binomial with parameters x and p.

(a) Find $E(Y)$.

(b) Find $\mathrm{Var}(Y)$.

(c) Let $Z = X - Y$ and $q = 1 - p$. Verify that

$$p_{Y,Z}(y,z) = \left(\frac{(p\lambda)^y}{y!} e^{-p\lambda} \right) \left(\frac{(q\lambda)^z}{z!} e^{-q\lambda} \right)$$

for non-negative integers y and z. This shows that Y and Z are independent Poisson random variables of rates $p\lambda$ and $q\lambda$, respectively.

Hint: For non-negative integers y and z, note that

$$\begin{aligned} p_{Y,Z}(y,z) &= P(Y = y,\ Z = z) \\ &= P(Y = y,\ X = y + z) \\ &= P(Y = y \mid X = y + z) P(X = y + z). \end{aligned}$$

Compute each probability appearing in the last line using the assumption.

Math 170E Lecture 1, Winter 2020	Homework 8	Due March 2, in class
Name:		**UID:**

4.4 Bivariate Distributions of the Continuous Type

1. Let

$$f_{X,Y}(x,y) = \frac{3}{16}xy^2, \qquad 0 \le x \le 2, \quad 0 \le y \le 2,$$

be the joint PDF of X and Y.

(a) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

(b) Are two random variables independent? Why or why not?

(c) Compute $P(X < Y)$.

(d) Compute $E(Y)$.

(e) Compute $\mathrm{Cov}(X,Y)$.
(You may use the fact that $E(X) = \frac{4}{3}$.)

Math 170E Lecture 1, Winter 2020	Homework 8	Due March 2, in class
Name:		**UID:**

2. Let X and Y be independent random variables of the continuous type. Then X and Y have the joint PDF:

$$f_{X,Y}(x,y) = f_X(x)f_Y(y).$$

Prove that events $\{a \le X \le b\}$ and $\{c \le Y \le d\}$ are independent, i.e.,

$$P(a \le X \le b,\ c \le Y \le d) = P(a \le X \le b)P(c \le Y \le d).$$

holds true, for any numbers a, b, c, d.

3. Let

$$f_{X,Y}(x,y) = \frac{3}{2}, \qquad 0 \le x \le 1, \quad x^2 \le y \le 1$$

be the joint PDF of X and Y.

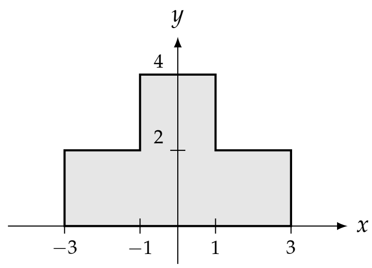
(a) Find $P(0 \le X \le \frac{1}{2})$.

(b) Find $P(\frac{1}{2} \le Y \le 1)$.

(c) Find $P(0 \le X \le \frac{1}{2},\ \frac{1}{2} \le Y \le 1)$.

(d) Are X and Y independent? Why or why not?

4. Let S be the shadowed region as in the figure below:



Suppose that (X, Y) have a uniform distribution over S, i.e., their joint PDF is given by

$$f_{X,Y}(x, y) = \frac{1}{16}, \qquad (x, y) \in S.$$

(a) Find the marginal PDF $f_X(x)$ of X.

(b) Determine the conditional PDF $f_{Y|X}(y|x)$.

(c) Calculate $E(Y \mid X = x)$. Does this make sense to you intuitively?

(d) Calculate $E(Y)$.

5. Let X have the uniform distribution $U(0,1)$, and let the conditional distribution of Y, given $X = x$, be $U(0,x)$. Find

$$P(X + Y \geq 1).$$

4.6 The Bivariate Normal Distribution

1. Let X and Y have a bivariate normal distribution with parameters $\mu_X = -3$, $\mu_Y = 10$, $\sigma_X^2 = 25$, $\sigma_Y^2 = 9$, and $\rho = 3/5$. Find

(a) The distribution of X.

(b) The conditional distribution of X, given $Y = 13$.

(c) $P(-5 < X < 5 \mid Y = 13)$, in terms of $\Phi(\cdot)$.

(d) The distribution of Y.

(e) The conditional distribution of Y, given $X = 2$.

(f) $P(7 < Y < 16 \mid X = 2)$, in terms of $\Phi(\cdot)$.

(g) $\text{Cov}(X, Y)$.