Math 170E Lecture 1, Winter 2020	Homework 6	Due February 14, in class
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- Some exercises are taken from R.V. Hogg, E. Tanis, D. Zimmerman, *Probability and Statistical Inference*, 10th Ed., Pearson.

3.1 Random Variables of the Continuous Type

1. Let X has a uniform distribution $U(a,b)$ for $a < b$. Do the following:

(a) Compute the mean $E(X)$.

(b) Compute the variance $\mathrm{Var}(X)$.

(c) Compute the moment generating function $M_X(t) = E[e^{tX}]$.
Hint: Treat the cases $t = 0$ and $t \neq 0$ separately.

2. Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} cx(1-x), & 0 < x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Do the following:

(a) Find the value of c such that $f_X(x)$ is indeed a PDF.

(b) Find $P(-0.5 < X < 0.3)$.

(c) Find the median of X.

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3. Let X be a continuous random variable with PDF

$$f_X(x) = \frac{c}{\sqrt{x}}, \qquad 0 < x < 1.$$

Do the following:

(a) Find the value of c such that $f_X(x)$ is indeed a PDF. Is this PDF bounded?

(b) Determine and sketch the graph of the CDF of X.

(c) Compute each of the following:

(i) $P(X > 0.5)$.

(ii) $P(X = 0)$.

(ii) The median of X.

(ii) The mean of X.

4. The logistic distribution is associated with the CDF

$$F_X(x) = \frac{1}{1+e^{-x}}, \qquad -\infty < x < \infty.$$

Find the PDF of ths logistic distribution and show that its graph is symmetric about $x = 0$.

5. Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} c(2+x), & -2 < x < -1, \\ c(2-x), & 1 < x < 2, \\ 0, & \text{elsewhere} \end{cases}$$

Do the following:

(a) Find the value of c such that $f_X(x)$ is indeed a PDF.

(b) Determine the CDF of X and sketch its graph.

(c) Find $P(X < 1.5)$.

(d) Find $m = \pi_{0.5}$ of X. Is it unique?

3.2 The Exponential, Gamma, and Chi-Square Distributions

6. Suppose that X has the MGF of the form

$$M_X(t) = \frac{1}{1-4t}, \qquad t < 1/4.$$

What are the PDF, the mean, and the variance of X?

7. Telephone calls arrive at a doctor's office according to a Poisson process on the average of two every 3 minutes. Let X denote the waiting time (in minutes) until the first call that arrives after 10 a.m.

(a) What is the PDF of X?

(b) Find $P(X > 2)$.

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8. Let X has an exponential distribution with mean θ, and let $c > 0$ be a positive constant. We will show that $Y = cX$ has an exponential distribution with mean $c\theta$ in two ways.

(a) Compute the MGF $M_Y(t)$ of Y by noting that

$$M_Y(t) = E[e^{tY}] = E[e^{ctX}] = M_X(ct),$$

where $M_X(\cdot)$ is the MGF of X. Determine the distribution of Y by identifying $M_Y(t)$ as the MGF of a familiar distribution.

(b) Find the CDF $F_Y(y)$ of Y by noting that

$$F_Y(y) = P(Y \leq y) = P(X \leq y/c) = F_X(y/c).$$

Determine the distribution of Y by identifying $F_Y(y)$ as the CDF of a familiar distribution.

9. In an experiment, alpha particle emissions of carbon-14 are counted by a Geiger counter each second. Suppose that the number of emissions follows a Poisson process with rate of 16 counts per second. Let W equal the time in seconds until the seventh count is made.

(a) Give the distribution of W.

(b) Find $P(W \leq 0.5)$.

Hint: Computing this by integrating the PDF will be excruciating. As in the class, interpret the event $\{W \leq 0.5\}$ in terms of the Poisson process involved.

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10. A bakery sells rolls in units of a dozen. The demand X (in 1000 units) for rolls has a gamma distribution with parameters $\alpha = 3$, $\theta = 0.5$, where θ is in units of days per 1000 units of rolls. It costs \$2 to make a unit that sells for \$5 on the first day when the rolls are fresh. Any leftover units are sold on the second day for \$1.

If w denotes the number of units to be made, then the above information suggests that the net profit $u(x)$ as a function of the demand x (in units of days per 1000 units of rolls) can be modeled by

$$u(x) = \frac{5000}{x} + \left(w - \frac{1000}{x}\right) - 2w,$$

where we conveniently ignored the case $1000/x \geq w$ for simplicity. Compute $E[u(X)]$ and determine the number of units that should be made to maximize the expected value of the net profit.

11. Suppose that X is $\chi^2(23)$. Find the following:

(a) $P(14.85 < X < 32.01)$.

Note: Table IV in Appendix B provides the value $\chi^2_\alpha(r)$ for some selected values of the parameters r and α.

(b) The mean and variance of X.

(c) $\chi^2_{0.05}(23)$ and $\chi^2_{0.95}(23)$.

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3.3 The Normal Distribution

12. If Z is $N(0,1)$, write each of the following probabilities in terms of $\Phi(z)$ and find its value.

(a) $P(0.47 < Z \leq 2.13)$

(b) $P(Z > -1.56)$

(c) $P(|Z| < 1.96)$

(d) $P(|Z| > 3)$

13. If X is $N(6, 5^2)$, write each of the following probabilities in terms of $\Phi(z)$ and find its value.

(a) $P(6 \leq X \leq 14)$

(b) $P(X > 15)$

(c) $P(|X - 6| < 5)$

(d) $P(|X - 6| > 12.4)$

14. If X is $N(650, 20^2)$, find a constant $c > 0$ such that

$$P(|X - 650| > c) = 0.05.$$

15. If the MGF of X is

$$M_X(t) = e^{166t + 200t^2}, \qquad -\infty < t < \infty,$$

find the mean and the variance of X.

16. If X is $N(0, \sigma^2)$, find the distribution of $W = X^2$.

17. Suppose that X is $N(\mu, \sigma^2)$. If a and b are constants such that $a \neq 0$, show that $Y = aX + b$ is $N(a\mu + b, a^2\sigma^2)$.

Hint: Compute the MGF of Y by noting that

$$M_Y(t) = E[e^{tY}] = E[e^{atX+bt}] = e^{bt}M_X(at),$$

where $M_X(\cdot)$ is the MGF of X. Identify $M_Y(t)$ as the MGF of a familiar distribution.