Homework 3

- Homework 3 is due January 24 in class.
- Exercises are taken from R.V. Hogg, E. Tanis, D. Zimmerman, *Probability and Statistical Inference*, 10th Ed., Pearson.
- The starred problems will not be graded.

2.1 Random Variables of the Discrete Type

Exercises

2. Let a chip be taken at random from a bowl that contains six white chips, three red chips, and one blue chip. Let the random variable $X = 1$ if the outcome is a white chip, let $X = 5$ if the outcome is a red chip, and let $X = 10$ if the outcome is a blue chip.

(a) Find the pmf of X.

(b) Graph the pmf as a line graph.

3. For each of the following, determine the constant c so that $p(x)$ satisfies the conditions of being a pmf for a random variable X, and then depict each pmf as a line graph:

(a) $p(x) = x/c, \qquad x = 1, 2, 3, 4.$

(b) $p(x) = cx, \qquad x = 1, 2, 3, \cdots, 10.$

(c) $p(x) = c(1/4)^x, \qquad x = 1, 2, 3, \cdots.$

(d) $p(x) = c(x+1)^2, \qquad x = 0, 1, 2, 3.$

(e) $p(x) = x/c, \qquad x = 1, 2, 3, \cdots, n.$

(f) $p(x) = \dfrac{c}{(x+1)(x+2)}, \qquad x = 0, 1, 2, 3, \cdots.$

Hint: In part (f), write $p(x) = \frac{1}{x+1} - \frac{1}{x+2}$.

7. Let a random experiment be the casting of a pair of fair six-sided dice and let X equal the minimum of the two outcomes.

(a) With reasonable assumptions, find the pmf of X.

(b) Draw a probability histogram of the pmf of X.

(c) Let Y equal the range of the two outcomes (i.e., the absolute value of the difference of the largest and the smallest outcomes). Determine the pmf $p_Y(y)$ of Y for $y = 0, 1, 2, 3, 4, 5$.

(d) Draw a probability histogram for $p_Y(y)$.

8. Let a random experiment consist of rolling a pair of fair dice, each having six faces, and let the random variable X denote the sum of the dice.

(a) With reasonable assumptions, determine the pmf $p(x)$ of X. *Hint:* Picture the sample space consisting of the 36 points (result on first die, result on second die), and assume that each has probability $1/36$. Find the probability of each possible outcome of X, namely, $x = 2, 3, 4, \cdots, 12$.

(b) Draw a probability histogram for $p(x)$.

10. A fair four-sided die has two faces numbered 0 and two faces numbered 2. Another fair four-sided die has its faces numbered 0, 1, 4, and 5. The two dice are rolled. Let X and Y be the respective outcomes of the roll. Let $W = X + Y$.

(a) Determine the pmf of W.
(b) Draw a probability histogram of the pmf of W.

11. Let X be the number of accidents per week in a factory. Let the pmf of X be

$$p(x) = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}, \qquad x = 0, 1, 2, \cdots.$$

Find the conditional probability of $X \geq 4$, given that $X \geq 1$.

2.2 Mathematical Expectation

Exercises

3. Let X be a discrete random variable with the Benford distribution introduced in Exercise 2.1-4, which has pmf $p_X(x) = \log_{10}\left(\frac{x+1}{x}\right)$, $x = 1, 2, \cdots, 9$. Show that $E(X) = 9 - \log_{10}(9!)$.

5. Let the random variable X be the number of days that a certain patient needs to be in the hospital. Suppose X has the pmf

$$p(x) = \frac{5-x}{10}, \qquad x = 1, 2, 3, 4.$$

If the patient is to receive \$200 from an insurance company for each of the first two days in the hospital and \$100 or each day after the first two days, what is the expected payment for the hospitalization?

6. Let the pmf of X be defined by $p(x) = 6/(\pi^2 x^2)$, $x = 1, 2, 3, \cdots$. Show that $E(X)$ does not exist in this case.

8. Let X be a random variable with support $\{1, 2, 3, 5, 15, 25, 50\}$, each point of which has the same probability $1/7$. Argue that $c = 5$ is the value that minimizes $h(c) = E(|X - c|)$. Compare c with the value of b that minimizes $g(b) = E[(X - b)^2]$.

10. In the casino game called **high–low**, there are three possible bets. Assume that \$1 is the size of the bet. A pair of fair six-sided dice is rolled and their sum is calculated. If you bet **low**, you win \$1 if the sum of the dice is $\{2, 3, 4, 5, 6\}$. If you bet **high**, you win \$1 if the sum of the dice is $\{8, 9, 10, 11, 12\}$. If you bet on $\{7\}$, you win \$4 if a sum of 7 is rolled. Otherwise, you lose on each of the three bets. In all three cases, your original dollar is returned if you win. Find the expected value of the game to the bettor for each of these three bets.

12 Suppose that a school has 20 classes: 16 with 25 students in each, three with 100 students in each, and one with 300 students, for a total of 1000 students.

(a) What is the average class size?
(b) Select a student randomly out of the 1000 students. Let the random variable X equal the size of the class to which this student belongs, and define the pmf of X.
(c) Find $E(X)$, the expected value of X. Does this answer surprise you?

Exercises not in the textbook

N1.* **(St. Petersburg paradox)** Consider the following game: Flip a fair coin repeatedly until the first time tail appears. Let X denote the number of flips. When the game is over, the player wins 2^X dollars.

(a) Show that the expected number of coin flips is $E(X) = 2$.
(b) Show that the expected winning of the game is $E(2^X) = \infty$.
(c) Does 1000 dollars sound like a fair price to pay for entering the game?

2.3 Special Mathematical Expectations

Exercises

1. Find the mean, variance, and index of skewness for the following discrete distributions:

(a) $p(x) = 1, \qquad x = 5.$
(b) $p(x) = 1/5, \qquad x = 1, 2, 3, 4, 5.$
(c) $p(x) = 1/5, \qquad x = 3, 5, 7, 9, 11.$
(d) $p(x) = x/6, \qquad x = 1, 2, 3.$
(e) $p(x) = (1 + |x|)/5, \qquad x = -1, 0, 1.$
(f) $p(x) = (2 - |x|)/4, \qquad x = -1, 0, 1.$

4. Let μ and σ^2 denote the mean and variance of the random variable X. Determine

$$E\left[\frac{X-\mu}{\sigma}\right] \qquad \text{and} \qquad E\left[\left(\frac{X-\mu}{\sigma}\right)^2\right].$$

5. Consider an experiment that consists of selecting a card at random from an ordinary deck of cards. Let the random variable X equal the value of the selected card, where Ace $= 1$, Jack $= 11$, Queen $= 12$, and King $= 13$. Thus, the space of X is $S = \{1, 2, 3, \cdots, 13\}$. If the experiment is performed in an unbiased manner, assign probabilities to these 13 outcomes and compute the mean μ of this probability distribution.

8. Let X equal the larger outcome when a pair of fair four-sided dice is rolled. The pmf of X is

$$p(x) = \frac{2x-1}{16}, \qquad x = 1, 2, 3, 4.$$

Find the mean, variance, and standard deviation of X.

10. Let X be a discrete random variable with pmf

$$p(x) = \begin{cases} 1/16, & x = -5, \\ 5/8, & x = -1, \\ 5/16, & x = 3. \end{cases}$$

Find the index of skewness of X. Is this distribution symmetric?