PROBLEM 1

Suppose that the annual income X, in tens of thousands of dollars, of a randomly selected person has a continuous distribution with the CDF:

$$F_X(x) = \begin{cases} 1 - \left(\frac{x}{4}\right)^{-2}, & x \geq 4, \\ 0, & x < 4. \end{cases}$$

(a) Find the PDF of X.

(b) Compute $E(X)$.

(a) The PDF $f_X(x)$ of X can be obatined by differentiating the CDF of X:

$$f_X(x) = F'_X(x) = \boxed{\begin{cases} 32x^{-3}, & x \geq 4, \\ 0, & x < 4. \end{cases}}$$

(b) We have

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,\mathrm{d}x = \int_4^{\infty} x \cdot \frac{32}{x^3}\,\mathrm{d}x = \left[-\frac{32}{x}\right]_4^{\infty} = \boxed{8}.$$

PROBLEM 2

Telephone calls arrive at a doctor's office according to a Poisson process of ten calls, on average, per hour. Let W denote the waiting time, in minutes, until the third call after 10 a.m.

Compute the value of $P(W \leq 30)$.

Let N be the number of telephone calls until 10:30 a.m. Then

$$\begin{aligned} P(W \leq 30) &= 1 - P(W > 30) \\ &= 1 - P(3^{\text{rd}} \text{ call did not arrive until 10:30 a.m.}) \\ &= 1 - P(\text{there are at most 2 calls until 10:30 a.m.}) \\ &= 1 - P(N \leq 2). \end{aligned}$$

Since N has the Poisson distribution with mean $30 \times \frac{10}{60} = 5$, we get

$$P(W \leq 30) = 1 - \sum_{k=0}^{2} \frac{5^k}{k!} e^{-5} = \boxed{1 - \frac{37}{2} e^{-5}}.$$

PROBLEM 3

The amount of time X (in minutes) a particular server takes to serve a customer has the CDF:

$$F_X(x) = \begin{cases} 1, & x \geq 30, \\ 1 - \frac{2}{5}e^{-x/5}, & 0 \leq x < 30, \\ 0, & x < 0. \end{cases}$$

(a) Find the probability that the server serve a given customer instantaneously, i.e., $X = 0$.

(b) Find the probability that the server takes more than 10 minutes to serve a given customer.

(a) At $x = 0$, the graph of F_X has a jump of size $\frac{3}{5}$. Since this jump size corresponds to $P(X = 0)$, we get

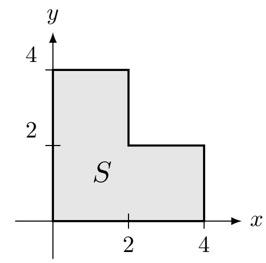
$$P(X = 0) = \boxed{\frac{3}{5}}.$$

(b) The desired probability is

$$P(X > 10) = 1 - P(X \leq 10) = 1 - F_X(10) = 1 - \left(1 - \frac{2}{5}e^{-2}\right) = \boxed{\frac{2}{5}e^{-2}}.$$

PROBLEM 4

Let S be the shaded region in the figure to the right. Suppose that (X, Y) have a uniform distribution over S, i.e., their joint PDF is given by



$$f_{X,Y}(x, y) = c, \qquad (x, y) \in S,$$

where c is a constant.

(a) Determine the value of c.

(b) Compute the value of $P(Y > 2X)$.

(a) We must have

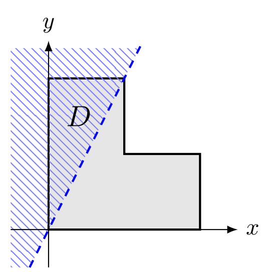
$$1 = \iint f_{X,Y}(x, y)\,\mathrm{d}x\mathrm{d}y = \iint_S c\,\mathrm{d}x\mathrm{d}y = c \cdot \text{area}(S).$$

Since area$(S) = 12$, we must have $c = \boxed{\frac{1}{12}}$.

(b) We have

$$P(Y > 2X) = \iint\limits_{y>2x} f_{X,Y}(x, y)\,\mathrm{d}x\mathrm{d}y.$$

To determine the bounds of the iterated integral, note that $y > 2x$ describes the region above the line $y = 2x$. See the hatched area in the figure below:



If we call this area by D, then

$$P(Y > 2X) = \iint_D \frac{1}{12}\,\mathrm{d}x\mathrm{d}y = \frac{1}{12} \cdot \text{area}(D) = \frac{4}{12} = \boxed{\frac{1}{3}}.$$

Alternatively, D may be described by $0 \le x \le 2$ and $2x < y \le 4$. So

$$P(Y > 2X) = \iint_D \frac{1}{12}\,\mathrm{d}x\mathrm{d}y = \int_0^2 \int_{2x}^4 \frac{1}{12}\,\mathrm{d}y\mathrm{d}x = \int_0^2 \frac{4 - 2x}{12}\,\mathrm{d}x = \boxed{\frac{1}{3}}.$$

PROBLEM 5

A fair six-sided die, with labels $1, 2, \ldots, 6$, is rolled 30 times. Let X and Y be the number of 1's and 2's, respectively.

(a) Find the conditional PMF of Y, given $X = x$, for $x = 0, \ldots, 30$.

(b) Find $E(Y \mid X)$. *(Hint: What is the conditional distribution of Y, given X?)*

(a) Since X is $b(30, \frac{1}{6})$, we have

$$p_X(x) = \frac{30!}{x!(30-x)!}\left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{30-x}, \qquad x = 0, \cdots, 30.$$

Also, since (X, Y) have the trinomial distribution with $n = 30$ and $p_1 = p_2 = \frac{1}{6}$, we get

$$p_{X,Y}(x, y) = \frac{30!}{x!y!(30-x-y)!}\left(\frac{1}{6}\right)^x \left(\frac{1}{6}\right)^y \left(\frac{4}{6}\right)^{30-x-y}, \qquad x, y \geq 0 \quad x + y \leq 30.$$

From this, for each $x = 0, \cdots, 30$ and for each $y = 0, \cdots, 30 - x$,

$$p_{Y|X}(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)} = \frac{\frac{\cancel{30!}}{\cancel{x!}y!(30-x-y)!}\cancel{\left(\frac{1}{6}\right)^x}\left(\frac{1}{6}\right)^y\left(\frac{4}{6}\right)^{30-x-y}}{\frac{\cancel{30!}}{\cancel{x!}(30-x)!}\cancel{\left(\frac{1}{6}\right)^x}\left(\frac{5}{6}\right)^{30-x}}$$

$$= \boxed{\binom{30-x}{y}\left(\frac{1}{5}\right)^y \left(\frac{4}{5}\right)^{30-x-y}}.$$

(b) The above computation shows that, for each $x = 0, \cdots, 30$, the conditional distribution of Y, given $X = x$, is $b(30 - x, \frac{1}{5})$. So we have

$$E(Y \mid X) = (30 - X) \cdot \frac{1}{5} = \boxed{6 - \frac{X}{5}}.$$