PROBLEM 1

Let X be a continuous random variable having the CDF:

$$F_X(x) = \begin{cases} \dfrac{x}{\sqrt{x^2+5}}, & 0 \le x, \\ 0, & x < 0. \end{cases}$$

Do the following:

(a) Compute the PDF of X.

(b) Find $P(-1 < X < 2)$.

Solution.

(a) The PDF of X can be obtained by differentiating the CDF:

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x) = \begin{cases} \dfrac{5}{(x^2+5)^{3/2}}, & 0 \le x, \\ 0, & x < 0. \end{cases}$$

(b) **1st Way:** Since the distribution of X is continuous,

$$\begin{aligned} P(-1 < X < 2) &= P(-1 < x \le 2) - P(X = 2) \\ &= F_X(2) - F_X(-1) - 0 \\ &= \boxed{\frac{2}{3}}. \end{aligned}$$

2nd Way: Alternatively,

$$P(-1 < X < 2) = \int_{-1}^{2} f_X(x)\,\mathrm{d}x = \int_{0}^{2} \frac{5}{(x^2+5)^{3/2}}\,\mathrm{d}x = \left[\frac{x}{\sqrt{x^2+5}}\right]_{x=0}^{x=2} = \boxed{\frac{2}{3}}.$$

□

PROBLEM 2

The weekly gravel demand X (in tons) follows the exponential distribution with mean $\theta = 8$. However, the owner of the gravel pit can produce at most only six tons of gravel per week. Let Y be the tons sold per week by the owner.

(a) Find the CDF of Y.

(b) Compute $P(3 \leq Y \leq 6)$.

Solution.

(a) Since $Y = \min\{X, 6\}$, the event $\{Y \leq y\}$ is the same as $\{X \leq y \text{ or } 6 \leq y\}$. So, depending on whether $y < 6$ or $y \geq 6$, this further simplifies to

$$\{Y \leq y\} = \begin{cases} \{X \leq y\}, & y < 6, \\ \{X \text{ is arbitrary}\}, & y \geq 6. \end{cases}$$

So we compute:

$$F_Y(y) = \begin{cases} P(X \leq y) = 0, & y < 0, \\ P(X \leq y) = 1 - e^{-y/8}, & 0 \leq y < 6, \\ 1, & 6 \leq y. \end{cases}$$

(b) **1st Way:** We have

$$\begin{aligned} P(3 \leq Y \leq 6) &= P(Y = 3) + P(3 < Y \leq 6) \\ &= P(Y = 3) + F_Y(6) - F_Y(3) \\ &= 0 + 1 - (1 - e^{-3/8}) = \boxed{e^{-3/8}}. \end{aligned}$$

2nd Way: Alternatively, investigating the jump sizes and derivative of $F_Y(y)$,

$$p_Y(y) = \begin{cases} e^{-6/8}, & y = 6 \\ 0, & \text{elsewhere}, \end{cases} \qquad \text{and} \qquad f_Y(y) = F'_Y(y) = \begin{cases} (1/8)e^{-y/8}, & 0 < y < 6, \\ 0, & \text{elsewhere}. \end{cases}$$

So it follows that

$$\begin{aligned} P(3 \leq Y \leq 6) &= \sum_{y:3 \leq x \leq 6} p_Y(y) + \int_3^6 f_Y(y)\,\mathrm{d}x \\ &= p_Y(6) + \int_3^6 (1/8)e^{-y/8}\,\mathrm{d}y \\ &= e^{-6/8} + (e^{-3/8} - e^{-6/8}) = \boxed{e^{-3/8}}. \end{aligned}$$

3rd Way: By arguing as in (a), we check that $\{3 \leq Y \leq 6\} = \{3 \leq X\}$. So

$$P(3 \leq Y \leq 6) = P(3 \leq X) = \boxed{e^{-3/8}}.$$

□

PROBLEM 3

Every smartphone returned to a repair center is classified according its needed repairs: (1) touch-screen, (2) battery, or (3) other. Past experience shows that 60% of broken smartphones need type 1 repairs, 30% need type 2 repairs, and 10% need type 3 repairs. Let X_i equal the number of type i repairs needed on a day in which four independently broken smartphones are returned.

(a) Find the probability that $X_1 = 1$ and $X_2 = 2$.

(b) Find $E(X_3)$.

Solution.

(a) X_1 and X_2 have the trinomial distribution with $n = 4$, $p_1 = \frac{6}{10}$, and $p_2 = \frac{3}{10}$. So

$$P(X_1 = 1, X_2 = 2) = \boxed{\frac{4!}{1!2!2!}\left(\frac{6}{10}\right)^1\left(\frac{3}{10}\right)^2\left(\frac{1}{10}\right)^1}.$$

(b) X_3 alone has the binomial distribution with $n = 4$ and $p_3 = \frac{1}{10}$. So

$$E(X_3) = (4)\left(\frac{1}{10}\right) = \boxed{\frac{4}{10}}.$$

□

PROBLEM 4

Let X and Y have the joint PDF:

$$f_{X,Y}(x,y) = 2, \qquad x > 0, \quad y > 0, \quad x + y < 1.$$

Do the following:

(a) Compute $\text{Cov}(X,Y)$.

(b) Are X and Y independent? Explain why or why not.

Solution.

(a) We have

$$\begin{aligned} E(X) &= \int_0^1 \int_0^{1-y} 2x \, \mathrm{d}x \mathrm{d}y = \int_0^1 (1-y)^2 \, \mathrm{d}y = \frac{1}{3}, \\ E(Y) &= \int_0^1 \int_0^{1-x} 2y \, \mathrm{d}y \mathrm{d}x = \int_0^1 (1-x)^2 \, \mathrm{d}x = \frac{1}{3}, \\ E(XY) &= \int_0^1 \int_0^{1-y} 2xy \, \mathrm{d}x \mathrm{d}y = \int_0^1 y(1-y)^2 \, \mathrm{d}y = \frac{1}{12}, \end{aligned}$$

and so,

$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{12} - \frac{1}{9} = \boxed{-\frac{1}{36}}.$$

(b) Since the covariance of X and Y is not zero, X and Y cannot be independent.

□

PROBLEM 5

Let X have the uniform distribution over the interval $[1, 2]$, and let Y, given $X = x$, have the normal distribution $\mathcal{N}(12x, 2x)$.

(a) Compute $E(XY)$.

(b) Compute $\mathrm{Var}(Y)$.

(You may use: the uniform distribution over $[a, b]$ has mean $\frac{a+b}{2}$ and variance $\frac{(b-a)^2}{12}$.)

Solution.

(a) Since $E(Y \mid X) = 12X$, it follows from the Law of Total Expectation that

$$\begin{aligned} E(XY) &= E[E(XY \mid X)] \\ &= E[XE(Y \mid X)] \\ &= 12E(X^2) \\ &= 12\big(\mathrm{Var}(X) + [E(X)]^2\big) \\ &= 12\left(\frac{1}{12} + \left(\frac{3}{2}\right)^2\right) = \boxed{28}. \end{aligned}$$

(b) Noting that $\mathrm{Var}(Y \mid X) = 2X$, the Law of Total Variance tells:

$$\begin{aligned} \mathrm{Var}(Y) &= E[\mathrm{Var}(Y \mid X)] + \mathrm{Var}(E(Y \mid X)) \\ &= E(2X) + \mathrm{Var}(12X) \\ &= 2 \cdot \left(\frac{3}{2}\right) + 12^2 \cdot \left(\frac{1}{12}\right) = \boxed{15}. \end{aligned}$$

□