PROBLEM 1

Suppose A and B are events such that $P(A) = 0.3$, $P(B) = 0.5$, and $P(A \cup B) = 0.7$. Do the following:

(a) Compute $P(A \cap B)$.

(b) Compute $P(A' \cap B')$.

Solution.

(a) From the identity $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, we get

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.3 + 0.5 - 0.7 = 0.1.$$

(b) By the De Morgan's Law, $A' \cap B' = (A \cup B)'$. So

$$P(A' \cap B') = P((A \cup B)') = 1 - P(A \cup B) = 1 - 0.7 = 0.3.$$

$\square$

PROBLEM 2

An urn contains five red and three white balls.

(a) Draw two balls from the urn at random without replacement. Find the probability that the selected balls are of different color.

(b) Flip a fair coin whose faces are marked 1 and 2. If k comes up, draw k balls from the urn at random. Find the probability that all of the balls drawn are red.

If you proceed by way of counting, clearly identify the counting method you are using.

Solution.

(a) Let R_1 and R_2 be events given by:

$$\begin{aligned} R_1 &= \{\text{the } 1^{\text{st}} \text{ draw is red}\}, \\ R_2 &= \{\text{the } 2^{\text{nd}} \text{ draw is red}\}. \end{aligned}$$

Then the desired probability is

$$\begin{aligned} &P(\text{the selected balls are of different color}) \\ &\quad = P(R_1 \cap R_2') + P(R_1' \cap R_2) \\ &\quad = P(R_1)P(R_2' \mid R_1) + P(R_1')P(R_2 \mid R_1') \\ &\quad = \frac{5}{8} \cdot \frac{3}{7} + \frac{3}{8} \cdot \frac{5}{7} = \frac{15}{28}, \end{aligned}$$

where we utilized the multiplication rule in the second step.

(b) Let R be the event that all the balls drawn are red. Then by the law of total probability,

$$\begin{aligned} P(R) &= P(1 \text{ comes up})P(R \mid 1 \text{ comes up}) + P(2 \text{ comes up})P(R \mid 2 \text{ comes up}) \\ &= \frac{1}{2} \cdot \frac{5}{8} + \frac{1}{2} \cdot \left(\frac{5}{8} \cdot \frac{4}{7}\right) = \frac{55}{112}. \end{aligned}$$

$\square$

PROBLEM 3

A medical test identifies a disease in 99% of cases when the patient is actually sick but also has a 1% rate of false positives – which means that the test comes out positive when the patient is actually not sick. The disease affects roughly 1% of the population. Decide whether the test is good by computing the probability that, given that the test comes out positive on a random person, the person is actually sick.

Solution. Write

$$\begin{aligned} D^+ &= \{\text{the patient is sick}\}, & T^+ &= \{\text{the test comes out positive}\} \\ D^- &= \{\text{the patient is not sick}\}, & T^- &= \{\text{the test comes out negative}\}. \end{aligned}$$

Then the assumptions translate to:

$$P(T^+ \mid D^+) = 0.99, \qquad P(T^+ \mid D^-) = 0.01, \qquad P(D^+) = 0.01.$$

Then by the Bayes' Theorem, the desired probability is computed by

$$\begin{aligned} P(D^+ \mid T^+) &= \frac{P(D^+)P(T^+ \mid D^+)}{P(D^+)P(T^+ \mid D^+) + P(D^-)P(T^+ \mid D^-)} \\ &= \frac{0.01 \cdot 0.99}{0.01 \cdot 0.99 + (1 - 0.01) \cdot 0.01} = 0.5. \end{aligned}$$

$\square$

PROBLEM 4

A random variable X has the moment generating function of the form

$$M(t) = \frac{1}{6}\big(e^{-t} + 3 + 2e^{t/2}\big),$$

Do the following:

(a) Find the value of $P(X = 0)$.

(b) Compute $\mathrm{Var}(X)$.

Solution.

(a) The MGF is already given in a form of $M(t) = p_1 e^{b_1 t} + p_2 e^{b_2 t} + \cdots$, and so, the probability $P(X = b)$ is the "coefficient" of the term involving e^{bt}. Now by writing

$$M(t) = \frac{1}{6} \cdot e^{(-1)\cdot t} + \frac{3}{6} \cdot e^{0\cdot t} + \frac{2}{6} \cdot e^{\frac{1}{2}t}$$

we read out that $P(X = 0) = \frac{3}{6}$.

(b) Knowing the MGF, the variance can be computed by $\mathrm{Var}(X) = M''(0) - M'(0)^2$. Since

$$M'(t) = -\frac{1}{6}e^{-t} + \frac{1}{6}e^{t/2} \qquad \text{and} \qquad M''(t) = \frac{1}{6}e^{-t} + \frac{1}{12}e^{t/2},$$

it follows that

$$\mathrm{Var}(X) = M''(0) - M'(0)^2 = \left(\frac{1}{6} + \frac{1}{12}\right) - \left(-\frac{1}{6} + \frac{1}{6}\right)^2 = \frac{1}{4}.$$

$\square$

PROBLEM 5

The number of flaws in a roll of fiber optics cable follows Poisson distribution. Suppose you know that having two flaws is twice as likely than having one flaw. Do as follows:

(a) Find the expected value of number of flaws in a roll.

(b) Compute the probability that there are at most three flaws in a roll given that there is at least one flaw in this roll.

Solution.

(a) Write X for the number of flaws in a roll and $\lambda = E(X)$. Then X has a Poisson distribution with parameter λ. Now the condition translates to the equality

$$P(X = 2) = 2P(X = 1),$$

which then becomes

$$\frac{\lambda^2}{2!}e^{-\lambda} = 2\lambda e^{-\lambda}.$$

Solving this in terms of λ gives $\lambda = 4$, which is the desired answer.

(b) The desired probability is $P(X \leq 3 \mid X \geq 1)$. Expanding this using the definition,

$$P(X \leq 3 \mid X \geq 1) = \frac{P(\{X \leq 3\} \cap \{X \geq 1\})}{P(X \geq 1)} = \frac{P(1 \leq X \leq 3)}{1 - P(X < 1)}.$$

Since X takes only non-negative integers as values,

$$\begin{aligned} P(1 \leq X \leq 3) &= \sum_{x=1}^{3} P(X = x) = \sum_{x=1}^{3} \frac{4^x}{x!}e^{-4} \\ &= \left(\frac{4}{1!} + \frac{4^2}{2!} + \frac{4^3}{3!}\right)e^{-4} = \frac{68}{3}e^{-4} \end{aligned}$$

and

$$P(X < 1) = P(X = 0) = e^{-4}.$$

Therefore

$$P(X \leq 3 \mid X \geq 1) = \frac{68e^{-4}/3}{1 - e^{-4}} = \frac{68}{3(e^4 - 1)}.$$

□