Polynomials in Free Variables

Roland Speicher
Universität des Saarlandes
Saarbrücken

joint work with Serban Belinschi, Tobias Mai, and Piotr Sniady

Goal: Calculation of Distribution or Brown Measure of Polynomials in Free Variables

Tools:

- Linearization
- Subordination
- Hermitization

We want to understand distribution of polynomials in free variables.

What we understand quite well is:

sums of free selfadjoint variables

So we should reduce:

$$\boxed{\text{arbitrary polynomial}} \longrightarrow \boxed{\textbf{sums} \text{ of } \textbf{selfadjoint} \text{ variables}}$$

This can be done on the expense of going over to operator-valued frame.

Let $\mathcal{B} \subset \mathcal{A}$. A linear map

$$E : \mathcal{A} \to \mathcal{B}$$

is a conditional expectation if

$$E[b] = b \qquad \forall b \in \mathcal{B}$$

and

$$E[b_1 a b_2] = b_1 E[a] b_2 \qquad \forall a \in \mathcal{A}, \quad \forall b_1, b_2 \in \mathcal{B}$$

An operator-valued probability space consists of $\mathcal{B} \subset \mathcal{A}$ and a conditional expectation $E : \mathcal{A} \to \mathcal{B}$

Consider an operator-valued probability space $E : \mathcal{A} \to \mathcal{B}$.

Random variables $x_i \in \mathcal{A}$ $(i \in I)$ are free with respect to E (or free with amalgamation over $\mathcal{B}$) if

$$E[a_1 \cdots a_n] = 0$$

whenever $a_i \in \mathcal{B}\langle x_{j(i)} \rangle$ are polynomials in some $x_{j(i)}$ with coefficients from $\mathcal{B}$ and

$$E[a_i] = 0 \quad \forall i \qquad \text{and} \qquad j(1) \neq j(2) \neq \cdots \neq j(n).$$

Consider an operator-valued probability space $E : \mathcal{A} \to \mathcal{B}$.

For a random variable $x \in \mathcal{A}$, we define the operator-valued Cauchy transform:

$$G(b) := E[(b - x)^{-1}] \qquad (b \in \mathcal{B}).$$

For $x = x^*$, this is well-defined and a nice analytic map on the operator-valued upper halfplane:

$$\mathbb{H}^+(B) := \{b \in B \mid (b - b^*)/(2i) > 0\}$$

Theorem (Belinschi, Mai, Speicher 2013): Let x and y be selfadjoint operator-valued random variables free over $\mathcal{B}$. Then there exists a Fréchet analytic map $\omega\colon \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^+(\mathcal{B})$ so that

$$G_{x+y}(b) = G_x(\omega(b)) \text{ for all } b \in \mathbb{H}^+(\mathcal{B}).$$

Moreover, if $b \in \mathbb{H}^+(\mathcal{B})$, then $\omega(b)$ is the unique fixed point of the map

$$f_b\colon \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^+(\mathcal{B}), \qquad f_b(w) = h_y(h_x(w) + b) + b,$$

and

$$\omega(b) = \lim_{n\to\infty} f_b^{\circ n}(w) \qquad \text{for any } w \in \mathbb{H}^+(\mathcal{B}).$$

where

$$\mathbb{H}^+(\mathcal{B}) := \{b \in \mathcal{B} \mid (b - b^*)/(2i) > 0\}, \qquad h(b) := \frac{1}{G(b)} - b$$

The Linearization Philosophy:

In order to understand polynomials in non-commuting variables, it suffices to understand matrices of **linear** polynomials in those variables.

- Voiculescu 1987: motivation
- Haagerup, Thorbjørnsen 2005: largest eigenvalue
- Anderson 2012: the selfadjoint version
  (based on Schur complement)

Consider a polynomial p in non-commuting variables x and y. A linearization of p is an $N \times N$ matrix (with $N \in \mathbb{N}$) of the form

$$\hat{p} = \begin{pmatrix} 0 & u \\ v & Q \end{pmatrix},$$

where

- u, v, Q are matrices of the following sizes: u is $1 \times (N-1)$; v is $(N-1) \times N$; and Q is $(N-1) \times (N-1)$

- each entry of u, v, Q is a polynomial in x and y, each of degree ≤ 1

- Q is invertible and we have

$$p = -uQ^{-1}v$$

Consider linearization of p

$$\hat{p} = \begin{pmatrix} 0 & u \\ v & Q \end{pmatrix} \qquad p = -uQ^{-1}v \qquad \text{and} \qquad b = \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix} \quad (z \in \mathbb{C})$$

Then we have

$$(b - \hat{p})^{-1} = \begin{pmatrix} 1 & 0 \\ -Q^{-1}v & 1 \end{pmatrix} \begin{pmatrix} (z-p)^{-1} & 0 \\ 0 & -Q^{-1} \end{pmatrix} \begin{pmatrix} 1 & -uQ^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (z-p)^{-1} & * \\ * & * \end{pmatrix}$$

and thus

$$G_{\hat{p}}(b) = \mathrm{id} \otimes \varphi((b - \hat{p})^{-1}) = \begin{pmatrix} \varphi((z-p)^{-1}) & \varphi(*) \\ \varphi(*) & \varphi(*) \end{pmatrix}$$

Note: $\hat{p}$ is the sum of operator-valued free variables!

Theorem (Anderson 2012): One has

- for each p there exists a linearization $\hat{p}$
  (with an explicit algorithm for finding those)

- if p is selfadjoint, then this $\hat{p}$ is also selfadjoint

Conclusion: Combination of linearization and operator-valued subordination allows to deal with case of selfadjoint polynomials.

Input: $\boldsymbol{p(x,y),\ G_x(z), G_y(z)}$

↓

Linearize $p(x,y)$ to $\hat{p} = \hat{x} + \hat{y}$

↓

$G_{\hat{x}}(b)$ out of $G_x(z)$ and $G_{\hat{y}}(b)$ out of $G_y(z)$

↓

Get $w(b)$ as the fixed point of the iteration
$w \mapsto G_{\hat{y}}(b + G_{\hat{x}}(w)^{-1} - w)^{-1} - (G_{\hat{x}}(w)^{-1} - w)$

↓

$G_{\hat{p}}(b) = G_{\hat{x}}(\omega(b))$

↓

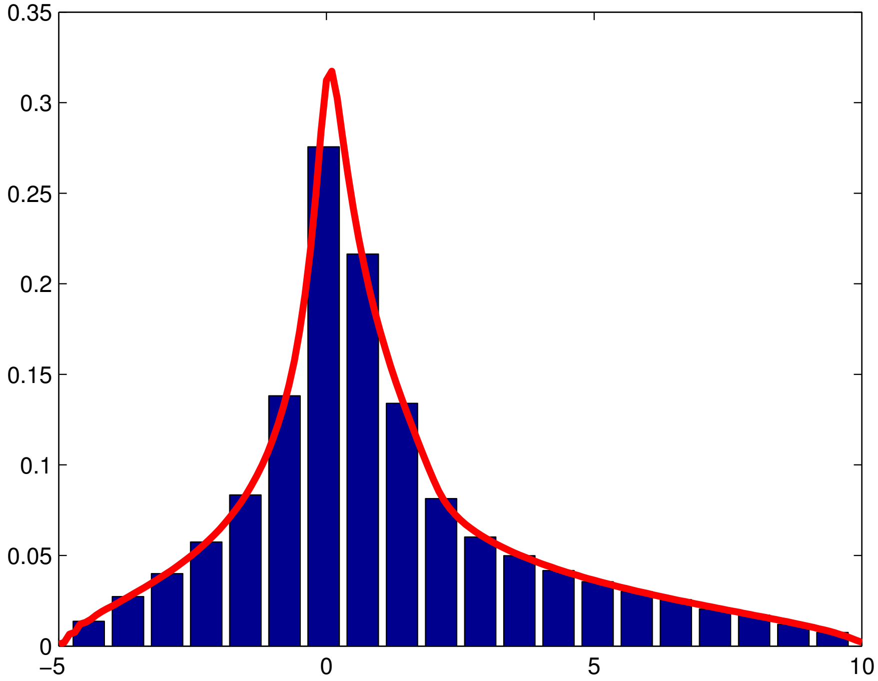
Recover $\boldsymbol{G_p(z)}$ as one entry of $\boldsymbol{G_{\hat{p}}(b)}$

Example: $p(x,y) = xy + yx + x^2$

p has linearization

$$\hat{p} = \begin{pmatrix} 0 & x & y + \frac{x}{2} \\ x & 0 & -1 \\ y + \frac{x}{2} & -1 & 0 \end{pmatrix}$$

$P(X, Y) = XY + YX + X^2$
for independent X, Y; X is Wigner and Y is Wishart



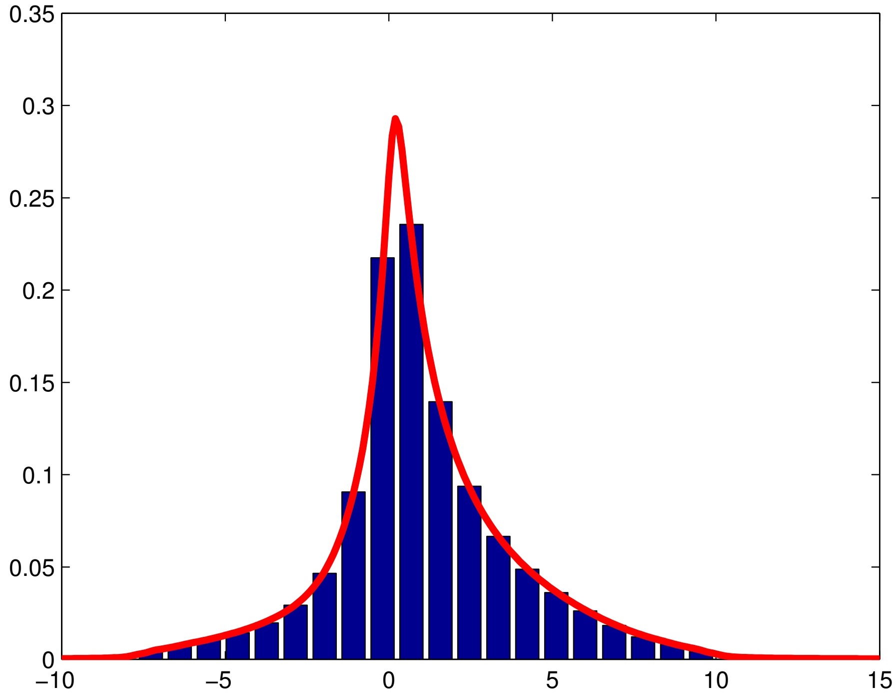
$p(x, y) = xy + yx + x^2$
for free x, y; x is semicircular and y is Marchenko-Pastur

Example: $p(x_1, x_2, x_3) = x_1x_2x_1 + x_2x_3x_2 + x_3x_1x_3$

p has linearization

$$\hat{p} = \begin{pmatrix} 0 & 0 & x_1 & 0 & x_2 & 0 & x_3 \\ 0 & x_2 & -1 & 0 & 0 & 0 & 0 \\ x_1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 & -1 & 0 & 0 \\ x_2 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_1 & -1 \\ x_3 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$P(X_1, X_2, X_3) = X_1X_2X_1 + X_2X_3X_2 + X_3X_1X_3$
for independent X_1, X_2, X_3; X_1, X_2 Wigner, X_3 Wishart



$p(x_1, x_2, x_3) = x_1x_2x_1 + x_2x_3x_2 + x_3x_1x_3$
for free x_1, x_2, x_3; x_1, x_2 semicircular, x_3 Marchenko-Pastur

What about non-selfadjoint polynomials?

For a measure on $\mathbb{C}$ its Cauchy transform

$$G_\mu(\lambda) = \int_{\mathbb{C}} \frac{1}{\lambda - z} d\mu(z)$$

is well-defined everywhere outside a set of $\mathbb{R}^2$-Lebesgue measure zero, however, it is analytic only outside the support of μ.

The measure μ can be extracted from its Cauchy transform by the formula (understood in distributional sense)

$$\mu = \frac{1}{\pi} \frac{\partial}{\partial \bar{\lambda}} G_\mu(\lambda),$$

Better approach by regularization:

$$G_{\epsilon,\mu}(\lambda) = \int_{\mathbb{C}} \frac{\bar{\lambda} - \bar{z}}{\epsilon^2 + |\lambda - z|^2} d\mu(z)$$

is well–defined for every $\lambda \in \mathbb{C}$. By sub-harmonicity arguments

$$\mu_\epsilon = \frac{1}{\pi} \frac{\partial}{\partial \bar{\lambda}} G_{\epsilon,\mu}(\lambda)$$

is a positive measure on the complex plane.

One has: $\lim_{\epsilon \to 0} \mu_\epsilon = \mu$ weak convergence

This can be copied for general (not necessarily normal) operators x in a tracial non-commutative probability space $(\mathcal{A}, \varphi)$.
Put

$$G_{\epsilon,x}(\lambda) := \varphi\Big((\lambda - x)^*\big((\lambda - x)(\lambda - x)^* + \epsilon^2\big)^{-1}\Big)$$

Then

$$\mu_{\epsilon,x} = \frac{1}{\pi}\frac{\partial}{\partial\bar{\lambda}} G_{\epsilon,\mu}(\lambda)$$

is a positive measure on the complex plane, which converges weakly for $\epsilon \to 0$,

$$\mu_x := \lim_{\epsilon \to 0} \mu_{\epsilon,x} \qquad \text{Brown measure of } x$$

Hermitization Method

For given x we need to calculate

$$G_{\epsilon,x}(\lambda) = \varphi\left((\lambda - x)^*\Big((\lambda - x)(\lambda - x)^* + \epsilon^2\Big)^{-1}\right)$$

Let

$$X = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in M_2(\mathcal{A}); \qquad \text{note: } X = X^*$$

Consider X in the $M_2(\mathbb{C})$-valued probability space with repect to $E = \text{id} \otimes \varphi : M_2(\mathcal{A}) \to M_2(\mathbb{C})$ given by

$$E\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right] = \begin{pmatrix} \varphi(a_{11}) & \varphi(a_{12}) \\ \varphi(a_{21}) & \varphi(a_{22}) \end{pmatrix}.$$

For the argument

$$\Lambda_\epsilon = \begin{pmatrix} i\epsilon & \lambda \\ \bar{\lambda} & i\epsilon \end{pmatrix} \in M_2(\mathbb{C}) \qquad \text{and} \qquad X = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix}$$

consider now the $M_2(\mathbb{C})$-valued Cauchy transform of X

$$G_X(\Lambda_\varepsilon) = E\Big[(\Lambda_\epsilon - X)^{-1}\Big] = \begin{pmatrix} g_{\epsilon,\lambda,11} & g_{\epsilon,\lambda,12} \\ g_{\epsilon,\lambda,21} & g_{\epsilon,\lambda,22} \end{pmatrix}.$$

One can easily check that

$$(\Lambda_\epsilon - X)^{-1} = \begin{pmatrix} -i\epsilon((\lambda - x)(\lambda - x)^* + \epsilon^2)^{-1} & (\lambda - x)((\lambda - x)^*(\lambda - x) + \epsilon^2)^{-1} \\ (\lambda - x)^*((\lambda - x)(\lambda - x)^* + \epsilon^2)^{-1} & -i\epsilon((\lambda - x)^*(\lambda - x) + \epsilon^2)^{-1} \end{pmatrix}$$

thus

$$g_{\epsilon,\lambda,12} = G_{\varepsilon,x}(\lambda).$$

So for a general polynomial we should

1. hermitize

2. linearise

3. subordinate

But: do (1) and (2) fit together???

Consider $p = xy$ with $x = x^*$, $y = y^*$.

For this we have to calculate the operator-valued Cauchy transform of

$$P = \begin{pmatrix} 0 & xy \\ yx & 0 \end{pmatrix}$$

Linearization means we should split this in sums of matrices in x and matrices in y.

Write

$$P = \begin{pmatrix} 0 & xy \\ yx & 0 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = XYX$$

$P = XYX$ is now a selfadjoint polynomial in the selfadjoint variables

$$X = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad Y = \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix}$$

XYX has linearization

$$\begin{pmatrix} 0 & 0 & X \\ 0 & Y & -1 \\ X & -1 & 0 \end{pmatrix}$$

thus

$$P = \begin{pmatrix} 0 & xy \\ yx & 0 \end{pmatrix}$$

has linearization

$$\begin{pmatrix} 0 & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & y & -1 & 0 \\ 0 & 0 & y & 0 & 0 & -1 \\ x & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ x & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y & 0 & 0 \\ 0 & 0 & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and we can now calculate the operator-valued Cauchy transform of this via subordination.

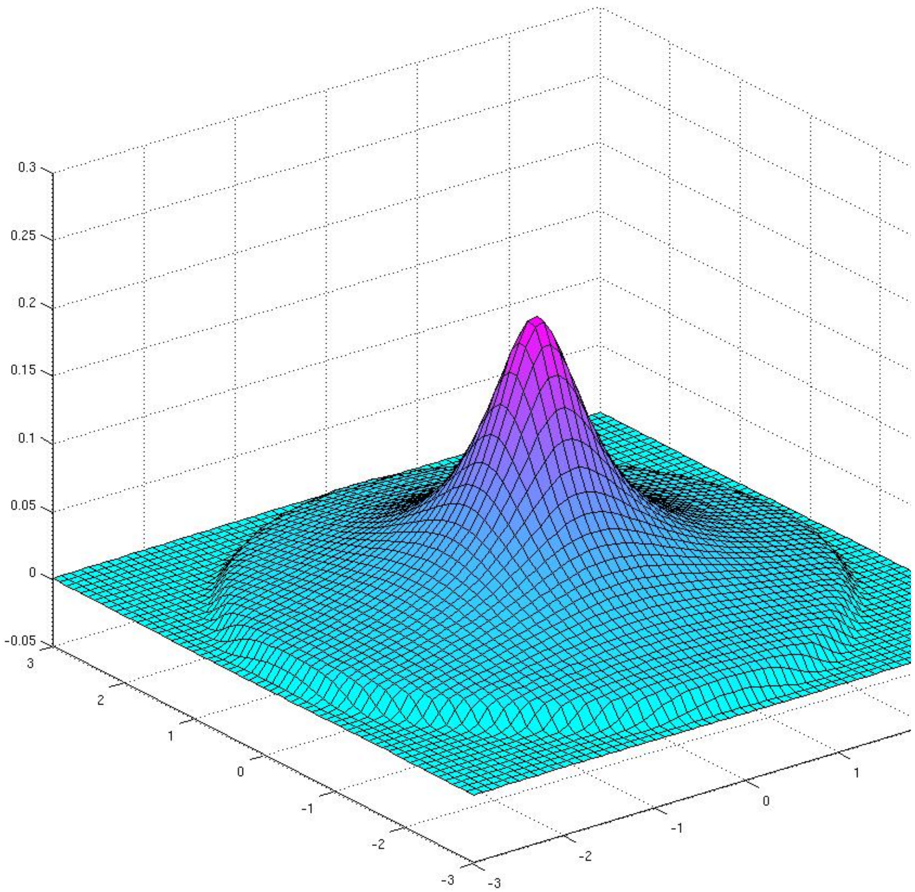
Does eigenvalue distribution of polynomial in independent random matrices converge to Brown measure of corresponding polynomial in free variables?

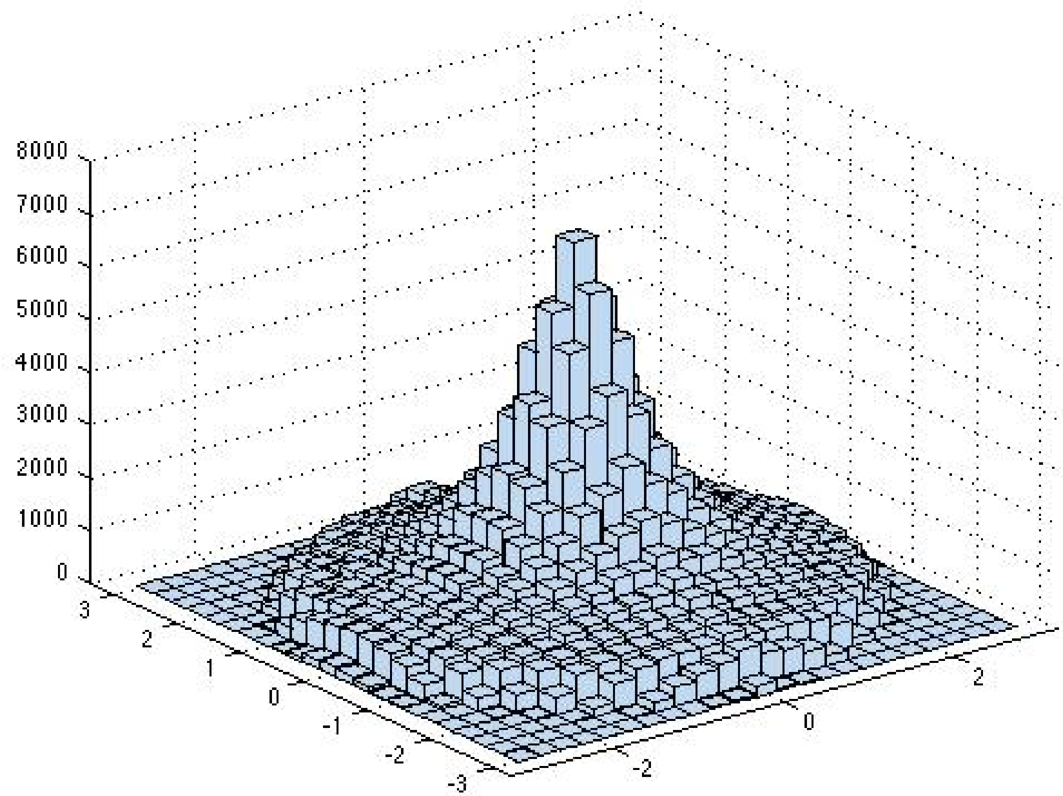
Conjecture: Consider m independent selfadjoint Gaussian (or, more general, Wigner) random matrices $X_N^{(1)},\dots,X_N^{(m)}$ and put

$$A_N := p(X_N^{(1)},\dots,X_N^{(m)}), \qquad x := p(s_1,\dots,s_m).$$

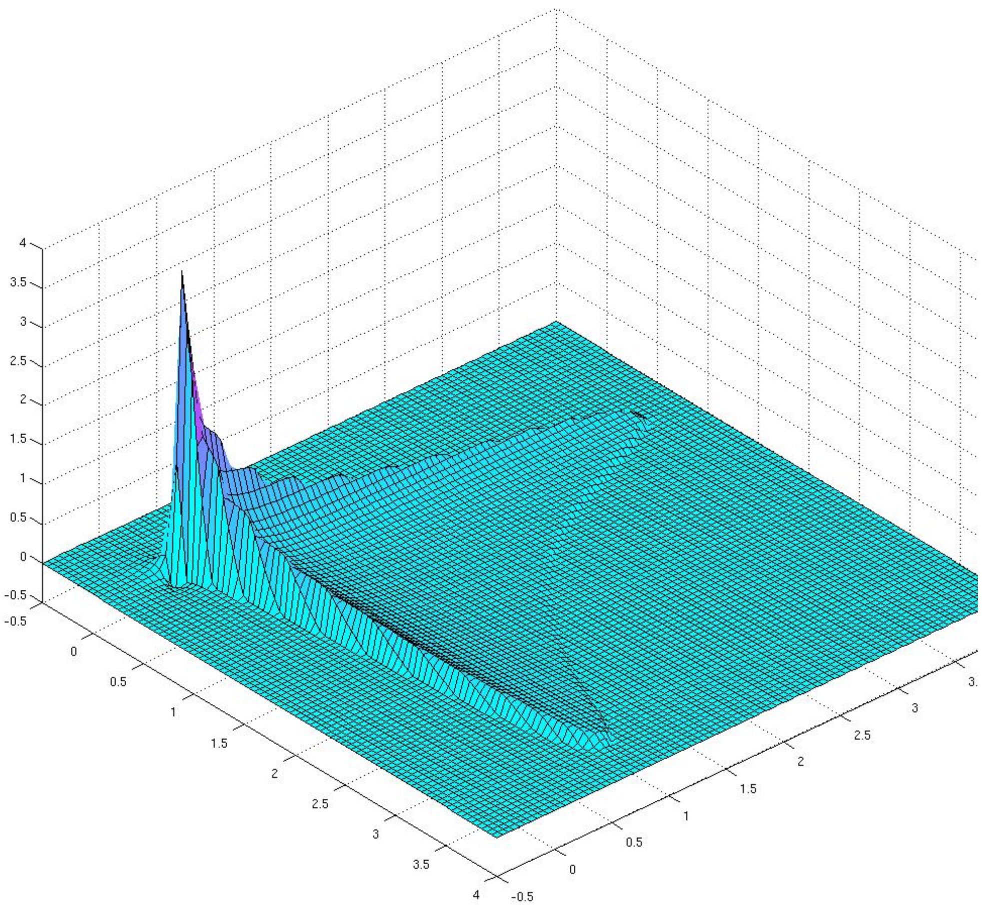
We conjecture that the eigenvalue distribution μ_{A_N} of the random matrices A_N converge to the Brown measure μ_x of the limit operator x.

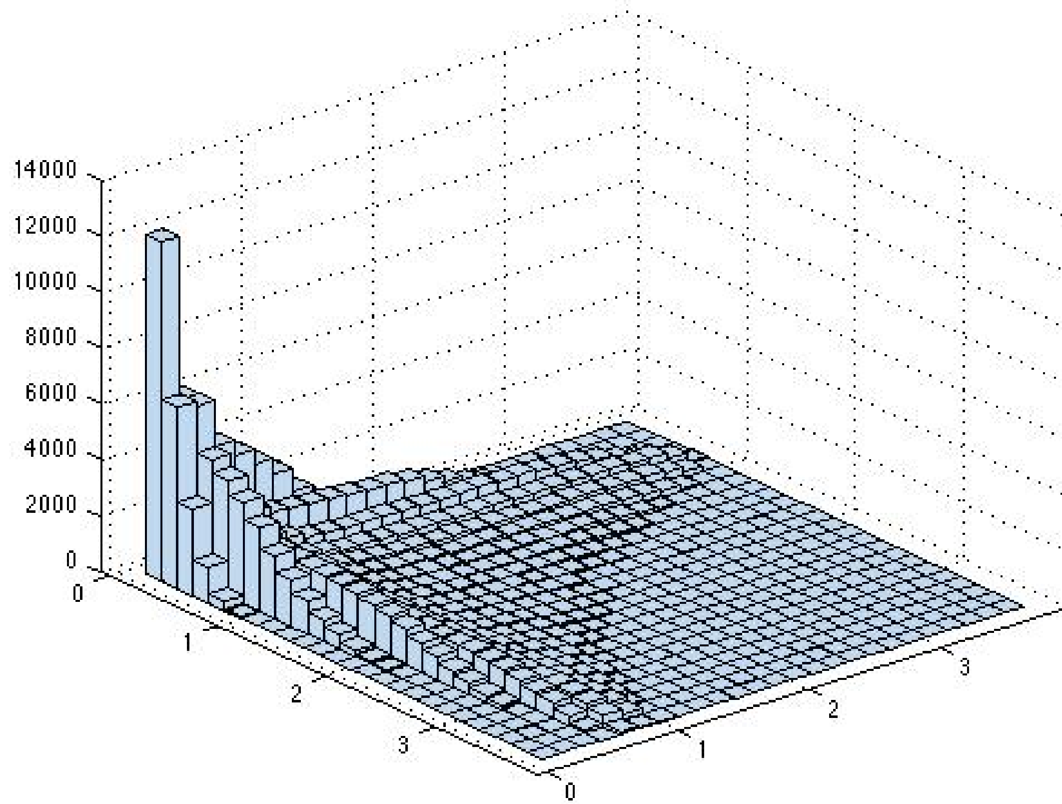
Brown measure of $xyz - 2yzx + zxy$ with x, y, z free semicircles





Brown measure of $x + iy$ with x, y free Poissons





Brown measure of $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1$