OUTLIER EIGENVALUES FOR DEFORMED I.I.D. RANDOM MATRICES

Charles Bordenave

CNRS & University of Toulouse

Joint work with Mireille Capitaine.

What this talk is about

Take

$$N = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & \dots & \end{pmatrix} \in M_n(\mathbb{R}),$$

and $U \in O(n)$.

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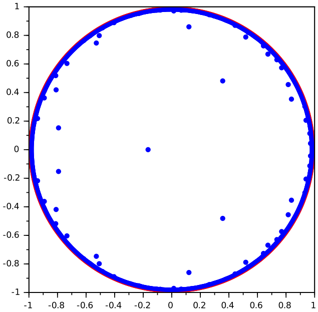
and $U \in O(n)$.

What are the eigenvalues of UNU^* *?*

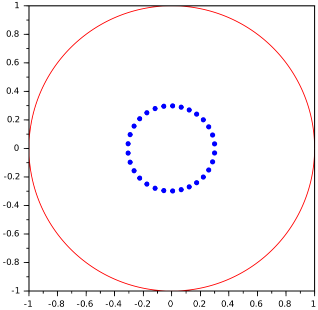
What this talk is about

Simulation for $n = 2000$ and U Haar distributed on $O(n)$.



What this talk is about

Simulation for $n = 30$ and U Haar distributed on $O(n)$.



Could a smoothed analysis explain this ?

von Neumann & Goldstine (1947), Edelman, Spielman & Teng (2001).

Deformed iid random matrices

Consider the non-hermitian random matrix model

$$M = \sigma Y + A,$$

where A is an $n \times n$ matrix, $\sigma > 0$ and

$$Y = \frac{X}{\sqrt{n}},$$

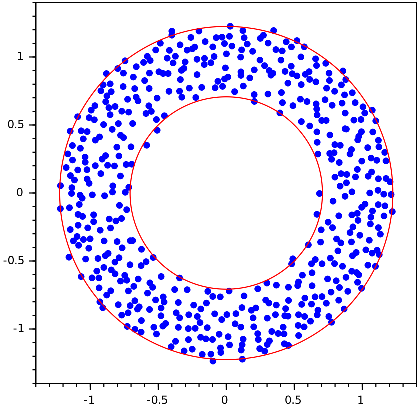
with $(X_{ij})_{i,j \geqslant 1}$ iid complex variables

$$\mathbb{E}X_{ij} = 0, \quad \mathbb{E}|X_{ij}|^2 = 1, \quad \mathbb{E}|X_{ij}|^4 < \infty.$$

Experimental mathematics

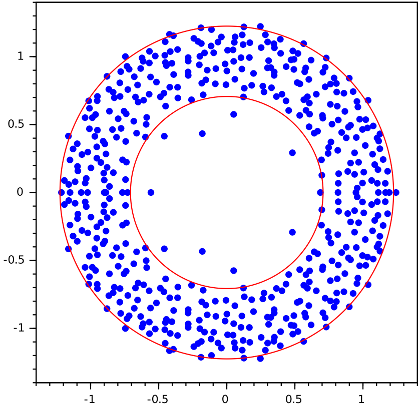
Circulant matrix

$$A = C_n = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \ddots & \ddots \\ 1 & 0 & \ldots & \end{pmatrix}, \quad n = 500, \quad \sigma^2 = 1/2.$$



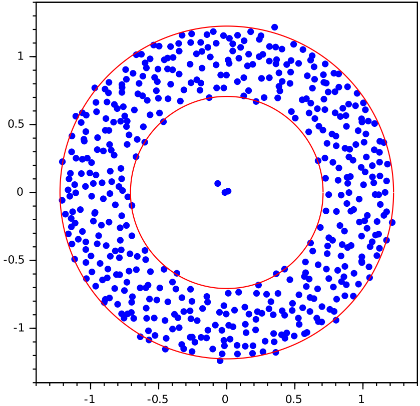
Nilpotent matrix

$$A = N_n = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & \cdots & \end{pmatrix}, \quad n = 500, \quad \sigma^2 = 1/2.$$



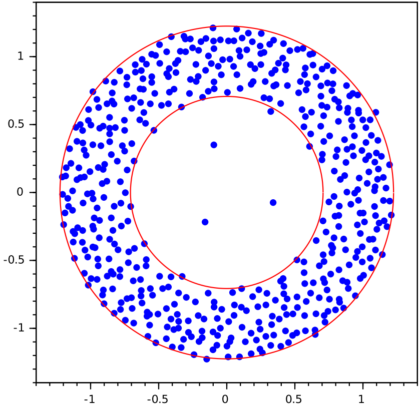
Outliers with diagonal Jordan blocks

$$A = \begin{pmatrix} C_{n-r} & 0 \\ 0 & 0_r \end{pmatrix}, \quad r = 3, \quad n = 500, \quad \sigma^2 = 1/2.$$



Outliers with full Jordan block

$$A = \begin{pmatrix} C_{n-r} & 0 \\ 0 & N_r \end{pmatrix}, \quad r = 3, \quad n = 500, \quad \sigma^2 = 1/2.$$



Convergence of spectral distributions

The two spectra

If $B \in M_n(\mathbb{C})$ has eigenvalues $\lambda_1(B), \ldots, \lambda_n(B)$, then

$$\mu_B = \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k(B)}$$

is the empirical distribution of the eigenvalues.

The two spectra

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$$\mu_B = \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k(B)}$$

is the empirical distribution of the eigenvalues.

The singular values of B will be denoted by

$$0 \leqslant s_n(B) \leqslant \ldots \leqslant s_1(B) = \|B\|.$$

We get

$$\mu_{BB^*} = \frac{1}{n} \sum_{k=1}^{n} \delta_{s_k^2(B)}.$$

Convergence of the spectral distributions

We will consider a sequence $A_n \in M_n(\mathbb{C})$ such that, as $n \to \infty$,

$$\|A_n\| = O(1),$$

and for all $z \in \mathbb{C}$, weakly,

$$\mu_{(A_n - z)(A_n - z)^*} \xrightarrow{w} \nu_z,$$

for some probability measure ν_z on $\mathbb{R}_+$.

Convergence of the spectral distributions

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for some probability measure ν_z on $\mathbb{R}_+$.

Example : A_n converges in $\star$-moments to an operator a in a tracial non-commutative probability space $(\mathcal{A}, \tau)$, i.e. for all $\varepsilon_\ell \in \{1, *\}$,

$$\frac{1}{n}\mathrm{Tr}(A_n^{\varepsilon_1} \dots A_n^{\varepsilon_k}) \to \tau(a^{\varepsilon_1} \dots a^{\varepsilon_k}),$$

then ν_z is the distribution of $(a - z)(a - z)^*$.

Convergence of the spectral distributions

Recall

$$M_n = \sigma Y_n + A_n.$$

Theorem (Śniady (2002), Dozier & Silverstein (2007), Tao & Vu (2010))

There exists a probability measure β *on* $\mathbb{C}$ *such that, a.s.*

$$\mu_{M_n} \xrightarrow{w} \beta.$$

For any $z \in \mathbb{C}$*, there exists a probability measure* μ_z *on* $\mathbb{R}_+$ *such that, a.s.*

$$\mu_{(M_n - z)(M_n - z)^*} \xrightarrow{w} \mu_z.$$

Brown's spectral measure

If A_n converges in $\star$-moments to an operator a. Set

$$b = \sigma c + a,$$

where c is a circular element free of a.

Then,

- μ_z is the distribution of $(b - z)(b - z)^*$,
- β is the Brown's spectral measure of b, i.e. in $\mathcal{D}'(\mathbb{C})$,

$$\beta = -\frac{1}{4\pi} \Delta \int_0^\infty \log(\lambda) d\mu_z(\lambda).$$

Haagerup & Larsen (2000), Biane & Lehner (2001), Bordenave, Chafaï & Caputo (2013), ...

Support of Brown's measure

We have

$$\mathrm{supp}(\beta) = \left\{ z \in \mathbb{C} : 0 \in \mathrm{supp}(\nu_z) \text{ or } \int \lambda^{-1} d\nu_z(\lambda) \geqslant \sigma^{-2} \right\},$$

provided that

$$\mathrm{supp}(\beta) = \{ z \in \mathbb{C} : 0 \in \mathrm{supp}(\mu_z) \}$$

(holds e.g. if a is a normal operator, always holds ?).

We will assume that the above formula holds and study the eigenvalues of M_n *in* $\mathbb{C} \backslash \mathrm{supp}(\beta)$.

No Outlier

Well-conditioned matrix

Theorem

Let $\Gamma \subset \mathbb{C} \backslash \mathrm{supp}(\beta)$ *be a compact set with continuous boundary.*

Assume that for all $z \in \Gamma$, *there exists* $\eta > 0$ *such that* $s_n(A_n - z) \geqslant \eta$ *for all* $n \gg 1$.

Then, a.s. for all $n \gg 1$, M_n *has no eigenvalue in* Γ.

In fact, more is true, for some $\delta > 0$ *a.s. for all* $n \gg 1$, $s_n(M_n - z) \geqslant \delta$ *for all* $z \in \Gamma$.

Well-conditioned matrix

Take the normal matrix

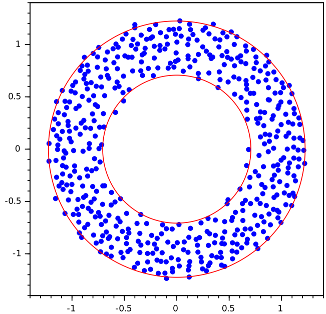
$$A_n = C_n.$$

The support of β is an annulus

$$\operatorname{supp}(\beta) = \left\{ z \in \mathbb{C} : \sqrt{(1-\sigma^2)_+} \leqslant |z| \leqslant \sqrt{1+\sigma^2} \right\}.$$

The singular values of $A_n - z$ are equal to

$$|e^{\frac{2i\pi k}{n}} - z| \geqslant |1 - |z||.$$

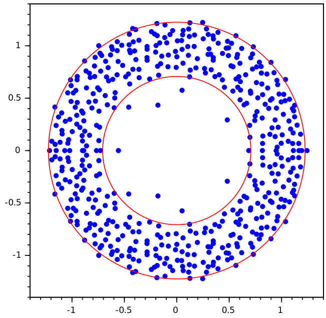


Badly-conditioned matrix

Take the nilpotent matrix

$$A_n = N_n.$$

Then β is unchanged but, if $|z| < 1$, $s_n(A_n - z) = o(1)$.



Stable Outliers

Related works

Decompose

$$A_n = A'_n + A''_n$$

with $\mathrm{rank}(A''_n) = r = O(1)$.

Case $A'_n = 0$ studied in *Tao (2013)*.

If A'_n is a Wigner matrix, contained in *O'Rourke & Renfrew (2013)*.

Finite rank perturbation of the single ring model :

$$U_n D_n V_n^* + A''_n,$$

with $D_n \geqslant 0$ diagonal, U_n, V_n independent Haar unitary, *Benaych-Georges & Rochet (2013)* and *Guionnet-Zeitouni (2012)* for $A''_n = 0$.

Well-conditioned decomposition

$$A_n = A'_n + A''_n \quad \text{with} \quad \mathrm{rank}(A''_n) = r = O(1).$$

Theorem

Let $\Gamma \subset \mathbb{C}\backslash\mathrm{supp}(\beta)$ *be a compact set with continuous boundary.*

Assume that for all $z \in \Gamma$*, there exists* $\eta > 0$ *such that* $s_n(A'_n - z) \geqslant \eta$ *for all* $n \gg 1$.

Assume that for some $\varepsilon > 0$*, for all* $n \gg 1$*,*

$$\min_{z \in \partial\Gamma} \left| \frac{\det(A_n - z)}{\det(A'_n - z)} \right| \geqslant \varepsilon.$$

Then, a.s. for $n \gg 1$*,* M_n *and* A_n *have the same number of eigenvalues in* Γ.

Stable outliers

An eigenvalue $\theta_n \to \theta$ of A_n is a stable outlier if for any $\delta > 0$, we can find $\Gamma \subset B(\theta, \delta)$ and $\varepsilon > 0$ such that for all $n \gg 1$,

$$\min_{z \in \partial\Gamma} \left| \frac{\det(A_n - z)}{\det(A'_n - z)} \right| \geqslant \varepsilon.$$

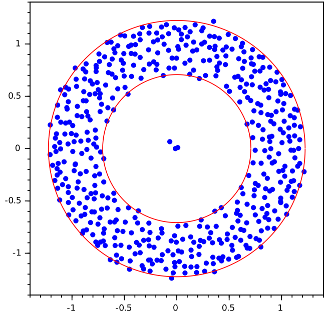
Counting multiplicities, to each stable outlier of A_n corresponds bijectively an eigenvalue at distance $o(1)$ in M_n.

Diagonal Jordan Blocks

$$A_n = \begin{pmatrix} C_{n-r} & 0 \\ 0 & 0_r \end{pmatrix} = \begin{pmatrix} C_{n-r} & 0 \\ 0 & I_r \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -I_r \end{pmatrix}.$$

If $|z| \geqslant \varepsilon^{1/r}(1+|z|)$,

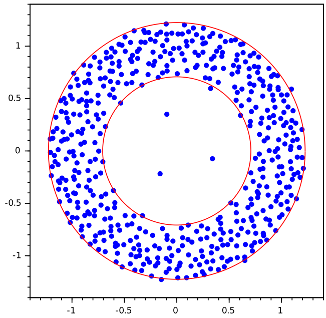
$$\left| \frac{\det(A_n - z)}{\det(A'_n - z)} \right| = \left| \prod_{k=1}^{r} \frac{-z}{1-z} \right| \geqslant \varepsilon.$$



Full Jordan Block

$$A_n = \begin{pmatrix} C_{n-r} & 0 \\ 0 & N_r \end{pmatrix} = \begin{pmatrix} C_{n-r} & 0 \\ 0 & I_r \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & N_r - I_r \end{pmatrix}.$$

Again, if $|z| \geqslant \varepsilon^{1/r}(1+|z|)$, $|\det(A_n - z)/\det(A'_n - z)| \geqslant \varepsilon$.



Huge fluctuations : $n = 500$ *is not enough to see the convergence of the* 3 *outliers to* 0 *!!*

Fluctuations of Stable Outliers

Diagonal Jordan Blocks

Assume $\theta_n \to \theta \in \mathbb{C}\backslash\mathrm{supp}(\beta)$ and

$$A_n = \begin{pmatrix} \theta_n I_r & 0 \\ 0 & \hat{A}_n \end{pmatrix}.$$

Theorem
Set $\varphi = \int \lambda^{-1} d\nu_\theta(\lambda)$*. Suppose that for some* $\eta > 0$ *and* $n \gg 1$*,* $s_n(\hat{A}_n - \theta) \geqslant \eta$ *and* $\frac{\mathbb{E}X_{11}^2}{n}\mathrm{Tr}\{(\hat{A}_n - \theta)^{-1}(\hat{A}_n^\top - \theta)^{-1}\} \to \psi$.

Then, a.s. for $n \gg 1$*,* M_n *has exactly* r *eigenvalues* $(\lambda_i)_{1\leqslant i\leqslant r}$ *in* $B(\theta, \eta/2)$*,*

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Theorem

Set $\varphi = \int \lambda^{-1} d\nu_\theta(\lambda)$. *Suppose that for some* $\eta > 0$ *and* $n \gg 1$, $s_n(\hat{A}_n - \theta) \geqslant \eta$ *and* $\frac{\mathbb{E}X_{11}^2}{n} \mathrm{Tr}\{(\hat{A}_n - \theta)^{-1}(\hat{A}_n^\top - \theta)^{-1}\} \to \psi$.

Then, a.s. for $n \gg 1$, M_n *has exactly* r *eigenvalues* $(\lambda_i)_{1 \leqslant i \leqslant r}$ *in* $B(\theta, \eta/2)$, *and*

$$\sqrt{n}\left((\lambda_1 - \theta_n), \ldots, (\lambda_r - \theta_n)\right)$$

converges in distribution towards the eigenvalues of

$$V = \sigma X_r + \sigma G \ \in M_r(\mathbb{C}),$$

where X_r *is independent of* G *with iid complex Gaussian entries given by,* $\mathbb{E}|Z_{ij}|^2 = \frac{\sigma^2 \varphi}{1 - \sigma^2 \varphi}$ *and* $\mathbb{E}Z_{ij}^2 = \frac{\sigma^2 (\mathbb{E}X_{11}^2)\psi}{1 - \sigma^2 \psi}$.

Full Jordan Block

Assume $\theta_n \to \theta \in \mathbb{C}\backslash\mathrm{supp}(\beta)$ and

$$A_n = \begin{pmatrix} P_n J_n P_n^{-1} & 0 \\ 0 & \hat{A}_n \end{pmatrix},$$

with $\|P_n - P\| \to 0$ for some $P \in \mathrm{GL}_r(\mathbb{C})$, and

$$J_n = \begin{pmatrix} \theta_n & 1 & & & \\ & \theta_n & 1 & & \\ & & \ddots & \ddots & \end{pmatrix} = \theta_n I_r + N_r.$$

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Theorem

Under the previous assumptions, a.s. for $n \gg 1$, M_n *has exactly* r *eigenvalues* $(\lambda_i)_{1\leqslant i\leqslant r}$ *in* $B(\theta, \eta/2)$, *and*

$$n^{1/2r}\left((\lambda_1 - \theta_n), \dots, (\lambda_r - \theta_n)\right)$$

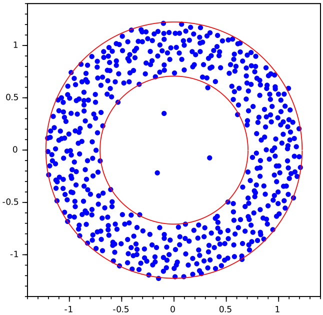
converges in distribution towards the roots of

$$z^r - e_r^* P^{-1} V P e_1.$$

Full Jordan Block

For $\hat{A}_n = 0$ and X complex Ginibre contained in *Benaych-Georges & Rochet (2013)*.

$$A_n = \begin{pmatrix} C_{n-r} & 0 \\ 0 & N_r \end{pmatrix}$$



For $n = 500$ *and* $r = 3$, $n^{-1/2r} \simeq 0.35$ *!!*

Unstable Outliers

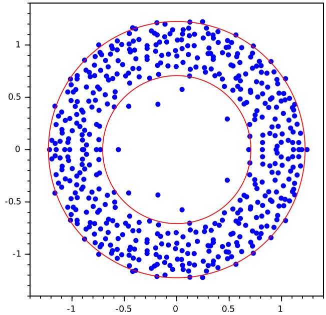
Nilpotent matrix

Take

$$N_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \end{pmatrix} = C_n - e_n e_1^*.$$

For any $|z| \leqslant 1 - \varepsilon$,

$$\left| \frac{\det(N_n - z)}{\det(C_n - z)} \right| = \frac{|z|^n}{|1 - z^n|} \leqslant \frac{(1-\varepsilon)^n}{1 - (1-\varepsilon)^n} = o(1).$$



Nilpotent matrix

In the orthonormal basis of eigenvectors of C_n, we get

$$C_n = U_n A'_n U_n^* \quad \text{and} \quad N_n = U_n(A'_n + A''_n)U_n^*$$

where

$$A'_n = \mathrm{diag}(e^{\frac{2i\pi}{n}}, \cdots, e^{\frac{2in\pi}{n}}) \quad \text{and} \quad A''_n = -f_n f_1^\top.$$

with $f_\ell = (e^{\frac{2i\pi\ell k}{n}}/\sqrt{n})_{1\leqslant k\leqslant n}$.

A''_n is a delocalized perturbation of A'_n.

Nilpotent matrix

Theorem
Let $A_n = A'_n + A''_n$ *be as above,* $0 < \sigma < 1$ *and assume that* $\mathbb{P}(|X_{ij}| \geqslant t) \leqslant \exp(-t^\kappa)$ *for some* $\kappa > 0$*. We set*

$$\varphi(z, w) = \frac{1}{1 - z\bar{w}}.$$

The point process of eigenvalues of M_n *in* $\mathring{B}(0, \sqrt{1-\sigma^2})$ *converges weakly to the zeros of the Gaussian analytic function* $g(z)$ *on* $\mathring{B}(0, \sqrt{1-\sigma^2})$ *with kernel given by,*

$$K(z, w) = \frac{\varphi(z, w)^2}{1 - \sigma^2 \varphi(z, w)}.$$

Gaussian analytic functions

Hough, Krishnapur, Peres, Virág (2009).

A Gaussian analytic function on $\Gamma \subset \mathbb{C}$ is a random analytic function g such that for every $z_1, \cdots, z_n$ in Γ,

$$(g(z_1), ..., g(z_n))$$

is a centered complex Gaussian vector with $\mathbb{E}g(z_i)g(z_j) = 0$.

The distribution of g is characterized by its kernel

$$K(z, w) = \mathbb{E}g(z)\bar{g}(w).$$

Edelman-Kostlan's formula : the intensity of zeros of g is

$$\frac{1}{2\pi} \Delta \log K(z, z).$$

Unstable outliers

We have a general convergence result for unstable outliers when A_n' is diagonal and for all $z \in \Gamma$,

$$\left|\frac{\det(A_n - z)}{\det(A_n' - z)}\right| = o\left(\frac{1}{\sqrt{n}}\right).$$

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In the unbounded component of $\mathbb{C} \backslash \mathrm{supp}(\beta)$, the above cannot hold.

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For A'_n normal with radial limiting ESD $\mu_{A'_n}$, there is a generic way to create unstable outliers in the bounded components of $\mathbb{C} \backslash \mathrm{supp}(\beta)$.

Large norm unstable outliers

Large unstable outliers

In the unbounded component of $\mathbb{C}\backslash\mathrm{supp}(\beta)$, it is possible to create unstable outliers when

$$A_n = A_n' + A_n''$$

with $\|A_n''\| \geqslant \sqrt{n}$,

Observed in *Rajan & Abbott (2006)* and *Tao (2011)*, for $A_n' = 0$ and a particular random choice of A_n''.

We have a general convergence result when A_n' diagonal, $A_n'' = \sqrt{n} v_n u_n^*$, and $u_n^*(z - A_n')^{-1} v_n$, $\|u_n\|_\infty$, $\|v_n\|_\infty$ of order $O(1/\sqrt{n})$.

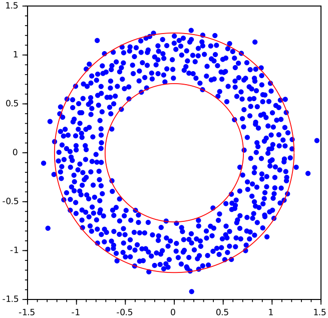
Large unstable outliers

The eigenvalues in $\mathbb{C}\backslash B(0,\sqrt{1+\sigma^2})$ of M_n when

$$A_n = \operatorname{diag}\left(e^{\frac{2i\pi}{n}},\cdots,e^{\frac{2in\pi}{n}}\right) + \sqrt{n} f_n f_1^\top$$

converge vaguely to the zeros of $1+\sigma g$ where g is the Gaussian analytic function with kernel

$$H(z,w) = \frac{\varphi(z,w)^2}{1+\sigma^2\varphi(z,w)}, \quad \text{with} \quad \varphi(z,w) = \frac{1}{1-z\bar{w}}.$$



Theorem
Assume that $\mathbb{P}(|X_{ij}| \geqslant t) \leqslant \exp(-t^\kappa)$ *for some* $\kappa > 0$. *Take*

$$M_n = Y_n + \theta_n v_n u_n^*,$$

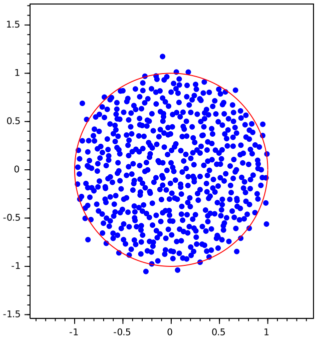
with $u_n^\top v_n = u_n^* v_n = 0$, $\theta_n \gg \sqrt{n}$ *and* $\|u_n\|_\infty, \|v_n\|_\infty = O(1/\sqrt{n})$.

The point process of eigenvalues of M_n *in* $\mathbb{C}\backslash B(0,1)$ *converges vaguely to the zeros of*

$$g(z) = \sum_{k \geqslant 0} \gamma_k z^{-k},$$

with γ_k *iid complex Gaussian variables with* $\mathbb{E}|\gamma_k|^2 = 1$ *and* $\mathbb{E}\gamma_k^2 = (\mathbb{E}X_{11}^2)^{k+1}$.

If $\mathbb{E}X_{11}^2 = 0$ then g is a GAF and its zeros is a determinantal point process, *Peres & Virág (2005)*.



$n = 500$ and $\theta_n = n^2$ (cropped image).

Ideas of proofs

Silvester's identity

Following *Benach-Georges & Rao (2011)* we use the identity, for $P, Q^\top \in M_{n,r}(\mathbb{C})$, $B \in \mathrm{GL}_n(\mathbb{C})$,

$$\det(B + PQ) = \det(B) \det(I_r + QB^{-1}P).$$

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$$\det(B + PQ) = \det(B)\det(I_r + QB^{-1}P).$$

Set $M_n' = \sigma Y_n + A_n'$,

$$R_n(z) = (zI_n - M_n')^{-1} \quad \text{and} \quad R_n'(z) = (zI_n - A_n')^{-1}.$$

For $r = 1$, if $A_n'' = v_n u_n^*$, we get

$$\frac{\det(zI_n - M_n)}{\det(zI_n - M_n')} = 1 - u_n^* R_n(z) v_n,$$

and

$$\frac{\det(zI_n - A_n)}{\det(zI_n - A_n')} = 1 - u_n^* R_n'(z) v_n.$$

Noise collapsing

Recall $M'_n = \sigma Y_n + A'_n$, $R_n(z) = (zI_n - M'_n)^{-1}$ and $R'_n(z) = (zI_n - A'_n)^{-1}$.

We prove that for any $z \in \mathbb{C} \backslash \text{supp}(\beta)$,

$$\boxed{u_n^* R_n(z) v_n = u_n^* R'_n(z) v_n + \frac{g_n(z)}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right).}$$

Hence,

$$\frac{\det(zI_n - M_n)}{\det(zI_n - M'_n)} = \frac{\det(zI_n - A_n)}{\det(zI_n - A'_n)} - \frac{g_n(z)}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right).$$

Rouché's Theorem

Let $U \subset \mathbb{C}$ is a bounded open connected set. We endow $\mathcal{H}(U)$ the set of analytic functions on U with the distance

$$d(f,g) = \sum_{j \geqslant 1} 2^{-j} \frac{\|f-g\|_{L^\infty(K_j)}}{1+\|f-g\|_{L^\infty(K_j)}}.$$

where K_j is an exhausting sequence of compact subsets of U. Then $\mathcal{H}(U)$ is a complete, separable metric space.

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Lemma

Let f_n be a tight sequence of random analytic functions which converges weakly to f for the finite dimensional convergence. If a.s. $f \not\equiv 0$ then the point process of zeros of f_n converges weakly to the point process of zeros of f.

More on noise collapsing

Recall $M'_n = \sigma Y_n + A'_n$, $R_n(z) = (zI_n - M'_n)^{-1}$ and $R'_n(z) = (zI_n - A'_n)^{-1}$. We prove, for any $z \in \mathbb{C} \backslash \mathrm{supp}(\beta)$,

$$u_n^* R_n(z) v_n = u_n^* R'_n(z) v_n + \frac{g_n(z)}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right).$$

We expand

$$R_n = R'_n + \sum_{k=1}^{\infty} (\sigma R'_n Y_n)^k R'_n.$$

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More on noise collapsing

Recall $M_n' = \sigma Y_n + A_n'$, $R_n(z) = (zI_n - M_n')^{-1}$ and $R_n'(z) = (zI_n - A_n')^{-1}$. We prove, for any $z \in \mathbb{C}\backslash\mathrm{supp}(\beta)$,

$$u_n^* R_n(z) v_n = u_n^* R_n'(z) v_n + \frac{g_n(z)}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right).$$

We expand

$$R_n = R_n' + \sum_{k=1}^{\infty} (\sigma R_n' Y_n)^k R_n'.$$

(i) the series is a.s. convergent in norm precisely when $z \in \mathbb{C}\backslash\mathrm{supp}(\beta)$,

(ii) a.s. $u^* P(B_1, \cdots, B_k, Y) v \to 0$ if P is a non trivial polynomial in Y and $B_\ell = O(1)$.

(iii) for the $1/\sqrt{n}$ fluctuation : functional CLT for $z \mapsto \sqrt{n}(u_n^* R_n'(z) Y_n(z))^k R_n'(z) v_n$ + tightness.

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- The finite n effects in non-hermitian random matrices are truly surprising.

- Many situations not covered by our work, e.g.
  * eigenvectors,
  * $\sigma \to 0$,
  * A_n'' has large rank,
  * other polynomials $P(A_1, \cdots, A_k, Y)$,
  * edge behavior,
  * $\ldots$

THANK YOU FOR YOUR ATTENTION!

Outlier eigenvalues for deformed i.i.d. random matrices, with Mireille Capitaine - `arXiv:1403.6001`.