Generalized Brownian motions with multiple processes

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Generalized Brownian motion

- The famous example of generalized Brownian motion is the q-semicircular operators on the q-Fock space of Bożejko and Speicher.
- Guță and Maassen developed a theory of generalized Brownian motion based on a symmetric Fock space construction.
- Guță did a partial generalization of the theory of Guță and Maassen to multiple processes indexed by some set $\mathcal{I}$.
- I'll discuss ideas that arise in the spirit of these works of Guță and Guță and Maassen.

Colored pair partitions

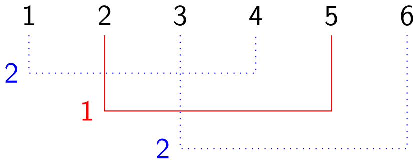
Fix an index set $\mathcal{I}$. It will be convenient to call the elements of $\mathcal{I}$ colors.

Definition

- Let $\mathcal{P}_2(2n)$ be the set of pair partitions of $[2n]$, i.e. partitions of $[2n]$ into blocks of size 2.
- $\mathcal{P}_2^{\mathcal{I}}(2n) := \{(\mathcal{V}, c) : \mathcal{V} \in \mathcal{P}_2(2n), c : \mathcal{V} \to \mathcal{I}\}$ is the set of $\mathcal{I}$-indexed pair partitions.
- Let $\mathcal{P}_2^{\mathcal{I}}(\infty) := \bigcup_{n=1}^{\infty} \mathcal{P}_2^{\mathcal{I}}(2n)$.

Colored pair partitions

We can draw colored pair partitions:



- $\mathcal{I} = \{1, 2\}$
- $\mathcal{V} = \{(1,4),(2,5),(3,6)\}$
- $c((1,4)) = c((3,6)) = 2$ and $c((2,5)) = 1$.
- solid red $\leftrightarrow 1 \in \mathcal{I}$
- dotted blue $\leftrightarrow 2 \in \mathcal{I}$

Symmetric Fock spaces

Suppose that for each $\mathbf{n} : \mathcal{I} \to \mathbb{N} \cup \{0\}$ with finitely many nonzero values, $V_{\mathbf{n}}$ is a complex Hilbert space with a unitary representation $U_{\mathbf{n}}$ of

$$S_{\mathbf{n}} := \prod_{b \in \mathcal{I}} S_{\mathbf{n}(b)},$$

If $\mathcal{H}$ is a complex Hilbert space, define a Fock space by

$$\mathcal{F}_V(\mathcal{H}) := \bigoplus_{\mathbf{n}} \frac{1}{\mathbf{n}!} V_{\mathbf{n}} \otimes_s \bigotimes_{b \in \mathcal{I}} \mathcal{H}^{\otimes \mathbf{n}(b)}$$

where $\otimes_s$ means the subspace of vectors fixed by the action of $S_{\mathbf{n}}$ given by $U_{\mathbf{n}} \otimes \tilde{U}_{\mathbf{n}}$, where $\tilde{U}_{\mathbf{n}}(\pi)$ permutes the vectors according to π, and $\mathbf{n}! := \prod_{b \in \mathcal{I}} n(a)!$ and $\frac{1}{\mathbf{n}!}$ refers to the inner product. Write $v \otimes_s f$ for the projection of $v \otimes f \in V_{\mathbf{n}} \otimes \bigotimes_{a \in \mathcal{I}} \mathcal{H}^{\otimes \mathbf{n}(a)}$ onto $V_{\mathbf{n}} \otimes_s \bigotimes_{a \in \mathcal{I}} \mathcal{H}^{\otimes \mathbf{n}(a)}$.

Creation and annihilation operators

Assume for $b \in \mathcal{I}$ we have $j_b : V_{\mathbf{n}} \to V_{\mathbf{n}+\delta_b}$ (where $\delta_b(b') = \delta_{b'b}$) with

$$j_b \cdot U_{\mathbf{n}}(\sigma) = U_{\mathbf{n}+\delta_b}(\iota_{\mathbf{n}}^{(b)}(\sigma)) \cdot j_b, \tag{1}$$

where $\iota_{\mathbf{n}}^{(b)}$ is the natural embedding $S_{\mathbf{n}} \hookrightarrow S_{\mathbf{n}+\delta_b}$. Define $\left(r_b^{(\mathbf{n})}\right)^*(h)$

$$\left(r_b^{(\mathbf{n})}\right)^*(h) : \bigotimes_{a\in\mathcal{I}} \mathcal{H}^{\otimes \mathbf{n}(a)} \to \bigotimes_{a\in\mathcal{I}} \mathcal{H}^{\otimes \mathbf{n}(a)+\delta_{a,b}}$$

acting as right creation operator on b-colored part $\mathcal{H}^{\otimes \mathbf{n}(b)}$.
The action on a vector $v \otimes_s \mathbf{f}$ of the creation operator $(a_b^{V,j})^*(h)$ is given by

$$(a_b^{V,j})^*(h) v_{\mathbf{n}} \otimes_s \mathbf{f} = \mathbf{n}(b)(j_b v_{\mathbf{n}}) \otimes_s (r_b^{\mathbf{n}})^*(h)\mathbf{f}.$$

The annihilation operator $a_b^{V,j}(h)$ is the adjoint of $(a_b^{V,j})^*(h)$. Denote by $\mathcal{C}_{V,j}(\mathcal{H})$ the $*$-algebra generated by the operators $a_b^{V,j}(f)$ and $(a_b^{V,j})^*(h)$ for $h \in \mathcal{H}$, and $b \in \mathcal{I}$.

Vacuum states of symmetric Fock spaces

Write

$$a_b^{V,j,e}(f) = \begin{cases} a_b^{V,j}(f), & \text{if } e = 1 \\ \left(a_b^{V,j}\right)^*(f), & \text{if } e = 2 \end{cases}$$

Theorem

Let $(U_{\mathbf{n}}, V_{\mathbf{n}})$ *be representations of* $S_{\mathbf{n}}$ *with maps* $j_b : V_{\mathbf{n}} \to V_{\mathbf{n}+\delta_b}$ *satisfying the intertwining relation. Let* $\xi_V \in V_0$ *be a unit vector and let* $\rho_{V,j}$ *be the vector state of* $\xi_V \otimes_s \Omega$ *on* $\mathcal{C}_{V,j}(\mathcal{H})$*. There is a* $\mathbf{t}_{V,j} : \mathcal{P}_2^{\mathcal{I}}(\infty) \to \mathbb{C}$ *such that*

$$\rho_{V,j}\left(\prod_{k=1}^{m} a_{b_k}^{V,j,e_k}(f_k)\right) = \sum_{(\mathcal{V},c)\in\mathcal{P}_2^{\mathcal{I}}(m)} \mathbf{t}_{V,j}((\mathcal{V},c)) \prod_{(l,r)\in\mathcal{V}} \langle f_l, f_r\rangle\, \delta_{b_l,b_r} B_{e_l e_r},$$

where $e_k \in \{1,2\}$ *and*

$$B := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Fock states

Remark

A state satisfying this pairing prescription is called a Fock state.

Example

The vacuum state on the algebra of q-creation and annihilation operators is a Fock state with $|\mathcal{I}| = 1$ and $\mathbf{t}(\mathcal{V}) = q^{\mathrm{cr}(\mathcal{V})}$.

Corollary

The restriction of $\rho_{V,j}$ to the algebra $\mathcal{A}_{V,j}(\mathcal{H})$ generated by the operators $\omega_b(e) := a_b^{V,j}(e) + \left(a_b^{V,j}\right)^*(e)$ is

$$\tilde{\rho}_{\mathbf{t}}\left(\prod_{k=1}^{m} \omega_{i_k}(f_k)\right) = \sum_{(\mathcal{V},c)\in\mathcal{P}_2^{\mathcal{I}}(m)} \mathbf{t}_{V,j}((\mathcal{V},c)) \prod_{(l,r)\in\mathcal{V}} \langle f_l, f_r\rangle\, \delta_{i_l,i_r}.$$

Spherical representations

- A spherical representation of $(G \times G, G)$ (where $G \hookrightarrow G \times G$ with the diagonal embedding) is an irreducible unitary representation of $G \times G$ with a non-zero G-fixed vector.
- The spherical representations of $(G \times G, G)$ are closely related to finite factor representations of G.
- Goal: construct a generalized Brownian motion associated to a spherical representation of $(S_\infty \times S_\infty, S_\infty)$.

Characters of S_∞

The characters of S_∞ are given by a famous theorem.

Theorem (Thoma 1964)

The normalized finite characters of S_∞ are given by the formula

$$\phi_{\alpha,\beta}(\sigma) = \prod_{m\geq 2}\left(\sum_{i=1}^{\infty}\alpha_i^m + (-1)^{m+1}\sum_{i=1}^{\infty}\beta_i^m\right)^{\rho_m(\sigma)} \tag{2}$$

where $\rho_m(\sigma)$ is the number of cycles of length m in the permutation σ, and $(\alpha_i)_{i=1}^{\infty}$ and $(\beta_i)_{i=1}^{\infty}$ are decreasing sequences of nonnegative real numbers such that $\sum_i \alpha_i + \sum_i \beta_i \leq 1$.

Vershik-Kerov representations of S_n

We'll discuss the case $\sum \alpha_n = 1$.

Notation

Fix a decreasing sequence (α_n) with $\sum \alpha_n = 1$. Define a measure μ on $\mathbb{N}$ by $\mu(n) = \alpha_n$. Let $\mathcal{X}_n = \prod_{i=1}^{n} \mathbb{N}$ with the product measure. Let S_n act on $\mathcal{X}_n$ by $\sigma(x_1, \ldots, x_n) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$. For $x, y \in \mathcal{X}_n$, say that $x \sim y$ if there exists $\sigma \in S_n$ such that $x = \sigma y$. Let $\tilde{\mathcal{X}}_n = \{(x, y) \in \mathcal{X}_n \times \mathcal{X}_n : x \sim y\}$. The Hilbert space $V_n^{(\alpha)}$ defined by

$$V_n^{(\alpha)} := \left\{ f : \tilde{\mathcal{X}}_n \to \mathbb{C} | \infty > \|f\|^2 = \int_{\mathcal{X}_n} \sum_{y \sim x} |f(x, y)|^2 dm_n^{(\alpha)}(x) \right\}$$

carries a unitary representation $U_n^{(\alpha)}$ of S_n given by

$$(U_n^{(\alpha)}(\sigma)h)(x, y) = h(\sigma^{-1}x, y).$$

Vershik-Kerov representations of S_∞

Denote by $\mathbf{1}_n$ the indicator function of the diagonal $\{(x,x)\} \subset \tilde{\mathcal{X}}_n$.

Theorem (Vershik, Kerov 1982)

On $V_n^{(\alpha)}$,

$$\left\langle U_n^{(\alpha)}(\sigma)\mathbf{1}_n, \mathbf{1}_n \right\rangle = \phi_\alpha(\sigma). \tag{3}$$

For $n = \infty$ we get the representation of S_∞ associated to ϕ_α in the convex hull of $\mathbf{1}_\infty$.

Generalized Brownian motions associated to factor representations of S_∞

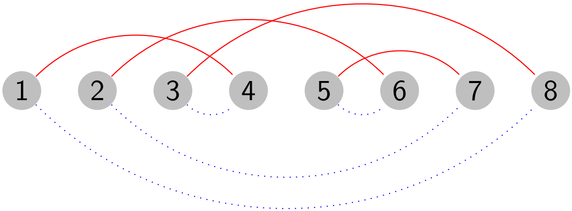
There is a natural embedding $j^\alpha : V_n^{(\alpha)} \to V_{n+1}^{(\alpha)}$ satisfying the necessary intertwining relation:

$$\delta_{((x_1,\ldots,x_n),(y_1,\ldots,y_n))} \mapsto \sum_{z\in\mathbb{Z}} \delta_{((x_1,\ldots,x_n,z),(y_1,\ldots,y_n,z))}. \tag{4}$$

Bożejko and Guţă used these representations of the S_n to construct generalized Brownian motions associated to factor representations of S_∞. They were able to compute the associated function on (one-colored) pair partitions.

Cycles of pair partitions

Bożejko and Guță introduced the notion of the cycle decomposition of a pair partition. Denote by $\hat{\mathcal{V}}$ the unique noncrossing pair partition such that the set of left points of $\mathcal{V} \in \mathcal{P}_2(\infty)$ and $\hat{\mathcal{V}}$ coincide. The cycle decomposition of $\mathcal{V} \in \mathcal{P}_2(2n)$ can be interpreted in terms of the multigraph $G_{\mathcal{V}}$ with vertices $[2n]$ and edge set $\mathcal{V} \coprod \hat{\mathcal{V}}$.



$G_{\mathcal{V}}$ is a union of vertex-disjoint cycles, a cycle of $\mathcal{V}$ is a set of the form $C \cap \mathcal{V}$ where C is a cycle of $G_{\mathcal{V}}$.

The formula of Bożejko and Guţă

Notation

Denote by $\rho_m(\mathcal{V})$ the number of cycles of $\mathcal{V}$ of length m.

Theorem (Bożejko, Guţă 2002)

The function on $\mathcal{P}_2(\infty)$ associated to the representations of Vershik and Kerov is given by

$$\mathbf{t}_{\alpha,\beta}(\mathcal{V}) = \prod_{m\geq 2}\left(\sum_{i=1}^{\infty}\alpha_i^m + (-1)^{m+1}\sum_{i=1}^{\infty}\beta_i^m\right)^{\rho_m(\mathcal{V})}. \tag{5}$$

What about spherical representations?

- The Hilbert space $V_n^{(\alpha)}$ consists of functions on certain pairs of n-tuples of natural numbers.
- With one color, S_n acted on $V_n^{(\alpha)}$ by permuting the left n-tuple.
- The group $S_n \times S_n$ acts on $V_n^{(\alpha)}$ with one copy permuting the left n-tuple and the other copy permuting the right n-tuple.
- For a symmetric Fock space indexed by $\mathcal{I} = \{1, 2\}$, we need representations of $S_{\mathbf{n}(1)} \times S_{\mathbf{n}(2)}$, even when $\mathbf{n}(1) \neq \mathbf{n}(2)$.
- Take $m = \max(\mathbf{n}(1), \mathbf{n}(2))$. Then we have a representation of $S_{\mathbf{n}(1)} \times S_{\mathbf{n}(2)}$ on $V_{\mathbf{n}}^{(\alpha)} := V_m^{(\alpha)}$ by restriction.
- If $\mathbf{n}(2) > \mathbf{n}(1)$ then $j_1^{(\alpha)} : V_{\mathbf{n}}^{(\alpha)} \to V_{\mathbf{n}+\delta_1}^{(\alpha)}$ is the identity map.

Associating a graph to a 2-colored pair partition

- Given $(\mathcal{V}, c) \in \mathcal{P}_2^{\mathcal{I}}(2n)$, we can define a multigraph $G_{(\mathcal{V},c)}$ with vertices $[2n]$.
- The pairs of $\mathcal{V}$ give half of the edges, and the color function c extends to a color function on all of the edges.
- The definition is quite technical, but an example might offer some intuition.

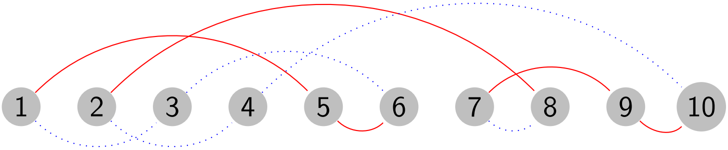
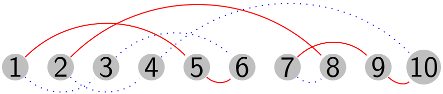


Figure : $G_{(\mathcal{V},c)}$ for $(\mathcal{V}, c) \in \mathcal{P}_2^{\mathcal{I}}(10)$ with $\mathcal{V} = \{(1,5),(2,8),(3,6),(4,10),(7,9)\}$ and $c((1,5)) = c((2,8)) = c((7,9)) = 1$ and $c((3,6)) = c((4,10)) = 2$.



- Right points correspond to creation operators, left points to annihilation operators.
- Applying 2-colored creation operator (corresponding to 10) to the vacuum vector gives an element of

$$V_{0,1}^{(\alpha)} \otimes_s \mathcal{H}^{\otimes 0} \otimes \mathcal{H}^{\otimes 1} = V_1^{(\alpha)} \otimes_s \mathcal{H}^{\otimes 0} \otimes \mathcal{H}^{\otimes 1}. \quad (6)$$

- Next applying a 1-colored creation operator (corresponding to 9) to the result gives an element of

$$V_{1,1}^{(\alpha)} \otimes_s \mathcal{H}^{\otimes 1} \otimes \mathcal{H}^{\otimes 1} = V_1^{(\alpha)} \otimes_s \mathcal{H}^{\otimes 1} \otimes \mathcal{H}^{\otimes 1}. \quad (7)$$

  We still have a function on 1-tuples!
- Edge between 9 and 10 keeps track of where elements are added to or removed from the tuples.

More about $G_{(\mathcal{V},c)}$

- In general, we can define a multigraph $G_{(\mathcal{V},c)}$ and extend the color function c to the edges.
- $G_{(\mathcal{V},c)}$ is a union of vertex-disjoint cycles, and each cycle has edges of both colors.
- In the one-color case we used cycle length, but here we consider the number of maximal monochrome paths in each cycle.
- The number of maximal monochrome paths in a cycle is always even. Denote by $\gamma_m(G_{(\mathcal{V},c)})$ the number of cycles of a 2-colored graph $G_{(\mathcal{V},c)}$ with $2m$ maximal monochrome paths. Equivalently, $\gamma_m(G_{(\mathcal{V},c)})$ is the number of cycles of $G_{(\mathcal{V},c)}$ having m maximal monochrome paths *of each color*.

The Fock state

Theorem (M 2014)

The vacuum state of the Fock space $\mathcal{F}_{V^{(\alpha,\beta)},j^{(\alpha,\beta)}}(\mathcal{H})$ *on the algebra of creation and annihilation operators is the Fock state arising from the function* $\mathbf{t}_{\alpha,\beta} : \mathcal{P}_2^{\mathcal{I}}(\infty) \to \mathbb{C}$ *given by*

$$\mathbf{t}_{\alpha,\beta}((\mathcal{V},c)) := \prod_{m\geq 2} \left(\sum_{i=1}^{\infty} \alpha_i^m + (-1)^{m+1} \sum_{i=1}^{\infty} \beta_i^m \right)^{\gamma_m(G_{(\mathcal{V},c)})} . \qquad (8)$$

Bibliography I

Stephen Avsec.
New examples of exchangeable noncommutative brownian motions.
In *Workshop on Analytic, Stochastic, and Operator Algebraic Aspects of Noncommutative Distributions and Free Probability*, Fields Institute, July 2013.

Marek Bożejko and Mădălin Guţă.
Functors of white noise associated to characters of the infinite symmetric group.
Comm. Math. Phys., 229(2):209–227, 2002.

Marek Bożejko and Roland Speicher.
An example of a generalized Brownian motion.
Comm. Math. Phys., 137(3):519–531, 1991.

Bibliography II

Marek Bożejko and Roland Speicher.
Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces.
Math. Ann., 300(1):97–120, 1994.

Marek Bożejko and Roland Speicher.
Interpolations between bosonic and fermionic relations given by generalized Brownian motions.
Math. Z., 222(1):135–159, 1996.

Mădălin Guţă.
The q-product of generalised Brownian motions.
In *Quantum probability and infinite dimensional analysis (Burg, 2001)*, volume 15 of *QP–PQ: Quantum Probab. White Noise Anal.*, pages 121–134. World Sci. Publ., River Edge, NJ, 2003.

Bibliography III

Mădălin Guţă and Hans Maassen.
Generalised Brownian motion and second quantisation.
J. Funct. Anal., 191(2):241–275, 2002.

Mădălin Guţă and Hans Maassen.
Symmetric Hilbert spaces arising from species of structures.
Math. Z., 239(3):477–513, 2002.

Franz Lehner.
Cumulants in noncommutative probability theory. III. Creation and annihilation operators on Fock spaces.
Infin. Dimens. Anal. Quantum Probab. Relat. Top., 8(3):407–437, 2005.

Bibliography IV

James A. Mingo and Alexandru Nica.
Crossings of set-partitions and addition of graded-independent random variables.
Internat. J. Math., 8(5):645–664, 1997.

A. Okoun′kov.
On representations of the infinite symmetric group.
Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 240(Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 2):166–228, 294, 1997.

Grigori Olshanski.
Unitary representations of (g, k)-pairs that are connected with the infinite symmetric group $s(\infty)$.
Leningrad Math. J, 1(4):983–1014, 1990.

Elmar Thoma.
Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe.
Math. Z., 85:40–61, 1964.

A. M. Vershik and S. V. Kerov.
Characters and factor-representations of the infinite unitary group.
Dokl. Akad. Nauk SSSR, 267(2):272–276, 1982.

Dan Voiculescu.
Symmetries of some reduced free product C^*-algebras.
In *Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983)*, volume 1132 of *Lecture Notes in Math.*, pages 556–588. Springer, Berlin, 1985.