Quantum Symmetric States

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[DKW] K. Dykema, C. Köstler, J. Williams, “Quantum symmetric states on universal free product C^*-algebras,” arXiv:1305.7293, to appear in Trans. AMS.

[DDM] Y. Dabrowski, K. Dykema, K. Mukherjee, “The simplex of tracial quantum symmetric states,” arXiv:1401.4692.

Classical Exchangeability

Definition

A sequence of (classical) random variables $x_1, x_2, \ldots$ is said to be *exchangeable* if

$$\mathbb{E}(x_{i(1)}x_{i(2)}\cdots x_{i(n)}) = \mathbb{E}(x_{\sigma(i(1))}x_{\sigma(i(2))}\cdots x_{\sigma(i(n))})$$

for every $n \in \mathbf{N}$, $i(1), \ldots, i(n) \in \mathbf{N}$ and every permutation σ of $\mathbf{N}$.

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for every $n \in \mathbf{N}$, $i(1), \ldots, i(n) \in \mathbf{N}$ and every permutation σ of $\mathbf{N}$.

That is, if the joint distribution of $x_1, x_2 \ldots$ is invariant under re-orderings.

De Finetti's Theorem

Theorem [de Finetti, 1937]

A sequence of random variables $x_1, x_2, \ldots$ is exchangeable if and only if the random variables are conditionally independent and identically distributed over its tail σ-algebra.

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The tail σ-algebra is the intersection of the σ-algebras generated by $\{x_N, x_{N+1}, \ldots\}$ as N goes to ∞.

Thus, the expectation $\mathbb{E}$ can be seen as an integral (w.r.t. a probability measure on the tail algebra) — that is, as a sort of convex combination — of expectations with respect to which the random variables $x_1, x_2, \ldots$ are independent and identically distributed (i.i.d.).

Symmetric states

Størmer extended this result to the realm of C*–algebras.

Definition

Consider the minimal tensor product $B = \bigotimes_1^\infty A$ of a C*–algebra A with itself infinitely many times. A state on B is said to be *symmetric* if it is invariant under the action of the group S_∞ by permutations of tensor factors.

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Note that the set $SS(A)$ of symmetric states on B is a closed, convex set in the set $S(B)$ of all states on B.

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Theorem [Størmer, 1969]

The extreme points of $SS(A)$ are the infinite tensor product states, i.e. those of the form $\otimes_1^\infty \phi$ for $\phi \in S(A)$ a state of A. Moreover, $SS(A)$ is a Choquet simplex, so every symmetric state on B is an integral w.r.t. a *unique* probability measure of infinite tensor product states.

The quantum permutation group of Shuzhou Wang [1998]

The quantum permutation group $A_s(n)$

$A_s(n)$ is the universal unital C*–algebra generated by a family of projections $(u_{i,j})_{1\leq i,j\leq n}$ subject to the relations

$$\forall i \ \sum_j u_{i,j} = 1 \quad \text{and} \quad \forall j \ \sum_i u_{i,j} = 1. \tag{1}$$

It is a compact quantum group (with comultiplication, counit and antipode).

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Abelianization of $A_s(n)$

The universal unital C*–algebra generated by *commuting* projections $\tilde{u}_{i,,j}$ satisfying the relations analogous to (1) is isomorphic to $C(S_n)$, the continuous functions of the permutation group S_n, with $\tilde{u}_{i,j} = 1_{\{\text{permutations sending } j\mapsto i\}}$. Thus, $C(S_n)$ is a quotient of $A_s(n)$ by a $*$-homomorphism sending $u_{i,j}$ to $\tilde{u}_{i,j}$.

Fully noncommutative version of permutation invariance

Invariance under quantum permutations

In a C*-noncommutative probability space (A, ϕ), the joint distribution of family $x_1, \ldots, x_n \in A$ is *invariant under quantum permtuations* if the natural coaction of $A_s(n)$ leaves the distribution unchanged. Concretely, this amounts to:

$$\phi(x_{i(1)} \cdots x_{i(k)})1 = \sum_{1 \leq j(1), \ldots, j(k) \leq n} u_{i(1),j(1)} \cdots u_{i(k),j(k)} \phi(x_{j(1)} \cdots x_{j(k)}) \in \mathbf{C}1 \subseteq A_s(n).$$

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Invariance under quantum permutations implies invariance under usual permuations

by taking the quotient from $A_s(n)$ onto $C(S_n)$.

Quantum exchangeable random variables and the tail algebra

Definition [Köstler, Speicher '09]

In a C*-noncommutative probability space, a sequence of random variables $(x_i)_{i=1}^{\infty}$ is *quantum exchangeable* if for every n, the joint distribution of $x_1, \ldots, x_n$ is invariant under quantum permutations.

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$$\mathcal{T} = \bigcap_{N=1}^{\infty} W^*(\{x_N, x_{N+1}, \ldots\}).$$

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Proposition [Köstler '10] (existence of conditional expectation)

Let $(x_i)_{i=1}^\infty$ be a quantum exchangeable sequence in a W*-noncommutative probability space $(\mathcal{M}, \phi)$ where ϕ is faithful and suppose $\mathcal{M}$ is generated by the x_i. Then there is a unique faithful, ϕ–preserving conditional expectation E from $\mathcal{M}$ onto $\mathcal{T}$.

Quantum exchangeable ⇔ free with amalgamation over tail algebra.

Theorem [Köstler, Speicher '09] (A noncommutative analogue of de Finneti's thoerem)

$(x_i)_{i=1}^{\infty}$ is a quantum exchangeable sequence if and only if the random variables are exchangeable and are free with respect to the conditional expectation E (i.e., with amalgamation over the tail algebra).

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Theorem [DK]

Given any countably generated von Neumann algebra $\mathcal{A}$ and any faithful state ψ on $\mathcal{A}$, there is a W*–noncommutative probability space $(\mathcal{M}, \phi)$ with ϕ faithful and with a sequence $(x_i)_{i=1}^{\infty}$ of random variables that is quantum exchangeable with respect to ϕ, and so that their tail algebra $\mathcal{T}$ is a copy of $\mathcal{A}$ so that $\phi\restriction_{\mathcal{T}}$ is equal to ψ.

Change in perspective

Generalize in the direction of C*-algebras, like Størmer did

Instead of considering individual random variables, we consider a unital C^*–algebra A and a state ψ on the universal unital free product C*–algebra $\mathfrak{A} = *_1^\infty A$, with corresponding embeddings $\lambda_i : A \to \mathfrak{A}$, $(i \geq 1)$.

Definition

A state ψ is symmetric if it is invariant under the action of the symmetric group on $\mathfrak{A}$.

Let ψ be a state on $\mathfrak{A}$ and let π_ψ be the GNS representation and $\mathcal{M}_\psi$ the von Neumann algebra generated by the image of π_ψ.

Proposition [DKW]

If ψ is symmetric, then there is a conditional expectation from $\mathcal{M}_\psi$ onto the *tail algebra* $\mathcal{T}_\psi = \bigcap_{N=1}^\infty W^*(\bigcup_{i=N}^\infty \pi_\psi \circ \lambda_i(A))$.

Definition [DKW]

A state ψ of $\mathfrak{A}$ is *quantum symmetric* if the $*$–homomorphisms λ_i are quantum exchangeable with respect to ψ, in the sense that, for all $n \in \mathbf{N}$, $a_1, \ldots, a_k \in A$ and $1 \leq i(1), \ldots, i(k) \leq n$,

$$\psi(\lambda_{i(1)}(a_1) \cdots \lambda_{i(k)}(a_k))1$$
$$= \sum_{1 \leq j(1), \ldots, j(k) \leq n} u_{i(1),j(1)} \cdots u_{i(k),j(k)} \psi(\lambda_{j(1)}(a_1) \cdots \lambda_{j(k)}(a_k))$$
$$\in \mathbf{C}1 \subseteq A_s(n).$$

Theorem [DKW]

Let ψ be a state of $\mathfrak{A}$. Then ψ is quantum symmetric if and only if it is symmetric and the images $\pi_\psi \circ \lambda_i(A)$ of the copies of A in the von Neumann algebra $\mathcal{M}_\psi$ are free with respect to E_ψ (i.e., with amalgamation over the tail algebra).

Remarks

- We don't require faithfulness of ψ on $\mathfrak{A}$, nor of $\hat{\psi}$ on $\mathcal{M}_\psi$, nor of E_ψ on $\mathcal{M}_\psi$.
- Our proof are similar to those in [Köstler, Speicher '09].
- Also Stephen Curran ['09] considered quantum exchangeability for sequences of $*$–homomorphisms of $*$-algebras and proved freeness with amalgamation; he did require faithfulness of a state, and used different ideas for his proofs.

Notation

Let $\mathrm{QSS}(A)$ denote the set of quantum symmetric states on $\mathfrak{A} = *_1^\infty A$. It is a closed, convex subset of the set of all states on $\mathfrak{A}$.

Goals

To investigate $\mathrm{QSS}(A)$ as a compact, convex subset of $S(\mathfrak{A})$, to characterize its extreme points and to study certain convex subsets:

- the *tracial quantum symmetric states* $\mathrm{TQSS}(A) = \mathrm{QSS}(A) \cap TS(\mathfrak{A})$
- the *central quantum symmetric states* $\mathrm{ZQSS}(A) = \{\psi \in \mathrm{QSS}(A) \mid \mathcal{T}_\psi \subseteq Z(\mathcal{M}_\psi)\}$
- the *tracial central quantum symmetric states* $\mathrm{ZTQSS}(A) = \mathrm{ZQSS}(A) \cap \mathrm{TQSS}(A)$.

Description of $\mathrm{QSS}(A)$ in terms of a single copy of A

There is a bijection $\mathcal{V}(A) \leftrightsquigarrow \mathrm{QSS}(A)$

where $\mathcal{V}(A)$ is the set of all quintuples $(\mathcal{B}, \mathcal{D}, E, \sigma, \rho)$ such that

- $1_{\mathcal{B}} \in \mathcal{D} \subseteq \mathcal{B}$ is a von Neumann subalgebra and $E : \mathcal{B} \to \mathcal{D}$ is a normal conditional expectation
- $\sigma : A \to \mathcal{B}$ is a unital $*$–homomorphism
- ρ is a normal state on $\mathcal{D}$ such that the state $\rho \circ E$ of $\mathcal{B}$ has faithful GNS representation
- $\mathcal{B} = W^*(\sigma(A) \cup \mathcal{D})$
- $\mathcal{D}$ is the smallest unital von Neumann subalgebra of $\mathcal{B}$ such that $E(d_0\sigma(a_1)d_1 \cdots \sigma(a_n)d_n) \in \mathcal{D}$ for all $a_1, \ldots, a_n \in A$ and all $d_0, \ldots, d_n \in \mathcal{D}$.

The bijection takes $(\mathcal{B}, \mathcal{D}, E, \sigma, \rho) \in \mathcal{V}(A)$, constructs the W*–free product $(\mathcal{M}, F) = (*_{\mathcal{D}})_1^\infty(\mathcal{B}, E)$ with amalgamation over $\mathcal{D}$, and yields the quantum symmetric state $\rho \circ E \circ (*_1^\infty \sigma)$ on $\mathfrak{A} = *_1^\infty A$.

Description of $\mathrm{QSS}(A)$ (2)

The correspondence $\mathcal{V}(A) \to \mathrm{QSS}(A)$

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Under the bijection:

from $(\mathcal{B}, \mathcal{D}, E, \sigma, \rho)$	$\mathcal{D}$	$\mathcal{M}$	$*_1^\infty \sigma$	F	$\rho \circ F$
from GNS rep of ψ	$\mathcal{T}_\psi$ (tail alg.)	$\mathcal{M}_\psi$	π_ψ	E_ψ (exp. onto tail alg.)	$\hat{\psi}$

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Technically, we need to let $\mathcal{V}(A)$ be the set of equivalence classes of quintuples, up to a natural notion of equivalence. Also, to avoid set theoretic difficulties we need to (and we can) restrict to $\mathcal{B}$ that are represented on some specific Hilbert space.

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Since the restriction of $\hat{\psi}$ to the tail algebra $\mathcal{T}_\psi$ is a normal state, this is equivalent to its support projection being a minimal projection of $\mathcal{T}_\psi$.

Central quantum symmetric states

Recall $\psi \in \mathrm{ZQSS}(A)$ means the tail algebra $\mathcal{T}_\psi$ lies in the center of $\mathcal{M}_\psi$, and $\mathrm{ZTQSS}(A)$ is the set of tracial ones.

Theorem [DKW]

- $\mathrm{ZQSS}(A)$ is a closed face of $\mathrm{QSS}(A)$ and is a Choquet simplex whose extreme points are the free product states:

$$\partial_e(\mathrm{ZQSS}(A)) = \{*_1^\infty \phi \mid \phi \in S(A)\}$$

- $\mathrm{ZTQSS}(A)$ is a closed face of $\mathrm{ZQSS}(A)$ and is a Choquet simplex whose extreme points are the free product traces:

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Choquet's theorem, then, implies that every element of $\mathrm{ZQSS}(A)$ is the barycenter of a unique probability measure on $\partial_e(\mathrm{ZQSS}(A))$, and likewise for $\mathrm{ZTQSS}(A)$.

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Choquet's theorem, then, implies that every element of $\mathrm{ZQSS}(A)$ is the barycenter of a unique probability measure on $\partial_e(\mathrm{ZQSS}(A))$, and likewise for $\mathrm{ZTQSS}(A)$. These are Bauer simplices, because their sets of extreme points are closed.

Tracial quantum symmetric states

Proposition [DKW]

$\mathrm{TQSS}(A)$ is in correspondence with the set of quintuples $(\mathcal{B}, \mathcal{D}, E, \sigma, \rho) \in \mathcal{V}(A)$ such that $\rho \circ E$ is a trace on $\mathcal{B}$ (which, then, must be faithful).

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In [DKW] we also found a (somewhat clumsy) characterization of the exteme points of $\mathrm{TQSS}(A)$.

Tracial quantum symmetric states (2)

A better characterization of extreme points of $\mathrm{TQSS}(A)$:

Theorem [DDM]

Let $\psi \in \mathrm{TQSS}(A)$ correspond to quintuple $(\mathcal{B}, \mathcal{D}, E, \sigma, \rho)$. (This implies $\mathcal{M}_\psi \cong (*_\mathcal{D})_1^\infty \mathcal{B}$ and the tail algebra $\mathcal{T}_\psi$ corresponds to $\mathcal{D}$.) Then the following are equivalent:

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Corollary [DDM]

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The key to the proof is to show $Z((*_\mathcal{D})_1^\infty \mathcal{B}) = Z(\mathcal{B}) \cap \mathcal{D}$.

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Thus, if A is separable and $A \neq \mathbf{C}$, then $\mathrm{TQSS}(A)$ is the Poulsen simplex (the unique metrizable simplex of more than one point whose extreme points are dense).

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Key idea of proof: perturb an arbitrary $\psi \in \mathrm{TQSS}(A)$ with a multiplicative free Brownian motion to get extreme points in $\mathrm{TQSS}(A)$.

Tracial quantum symmetric states (3)

Let $\psi \in \mathrm{TQSS}(A)$, and let $(\mathcal{B}, \mathcal{D}, E, \sigma, \rho)$ be the corresponding quintuple. (Thus, we have $\sigma : A \to \mathcal{B}$, $(\mathcal{M}_\psi, E_\psi) = (*_\mathcal{D})_1^\infty(\mathcal{B}, E)$, and the tail algebra is $\mathcal{D}$.)

Let $(U_t)_{t\geq 0}$ be a multiplicative free Brownian motion in $L(F_\infty)$, let $\widetilde{\mathcal{B}} = \mathcal{B} * L(F_\infty)$ and let $\sigma_t(\cdot) = U_t^* \sigma(\cdot) U_t$.

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By the free L^∞ Burkholder–Gundy inequality [Biane, Speicher '98], $\lim_{t\to 0^+} \|U_t - 1\| = 0$.

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We let $\widetilde{E} = E \circ E_{\mathcal{B}}^{\widetilde{\mathcal{B}}} : \widetilde{\mathcal{B}} \to \mathcal{D}$, where $E_{\mathcal{B}}^{\widetilde{\mathcal{B}}}$ is the canonical conditional expectation from $\widetilde{\mathcal{B}}$ onto $\mathcal{B}$. We let

$$(\widetilde{\mathcal{M}}, \widetilde{F}) = (*_{\mathcal{D}})_1^\infty (\widetilde{\mathcal{B}}, \widetilde{E})$$

and consider the state $\psi_t = \rho \circ \widetilde{F} \circ (*_1^\infty \sigma_t)$ on $\mathfrak{A}$.

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Using freeness, we have $\psi_t \in \mathrm{TQSS}(A)$, and using $U_t \to 1$, we have $\psi_t \to \psi$ in weak* topology as $t \to 0$.

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Using freeness, we have $\psi_t \in \mathrm{TQSS}(A)$, and using $U_t \to 1$, we have $\psi_t \to \psi$ in weak* topology as $t \to 0$.

We show that the tail algebra of ψ_t is a subalgebra of $\mathcal{D}$.

Tracial quantum symmetric states (4)

Recall, $\psi \leftrightsquigarrow (\mathcal{B}, \mathcal{D}, E, \sigma, \rho)$.

We have $\widetilde{\mathcal{B}} = \mathcal{B} * L(\mathbf{F}_\infty)$ and $\sigma_t(\cdot) = U_t^* \sigma(\cdot) U_t$.

We let $\widetilde{E} = E \circ E_{\mathcal{B}}^{\widetilde{\mathcal{B}}} : \widetilde{\mathcal{B}} \to \mathcal{D}$, where $E_{\mathcal{B}}^{\widetilde{\mathcal{B}}}$ is the canonical conditional expectation from $\widetilde{\mathcal{B}}$ onto $\mathcal{B}$. We let

$$(\widetilde{\mathcal{M}}, \widetilde{F}) = (*_{\mathcal{D}})_1^\infty (\widetilde{\mathcal{B}}, \widetilde{E})$$

and consider the state $\psi_t = \rho \circ \widetilde{F} \circ (*_1^\infty \sigma_t)$ on $\mathfrak{A}$.

Using freeness, we have $\psi_t \in \mathrm{TQSS}(A)$, and using $U_t \to 1$, we have $\psi_t \to \psi$ in weak* topology as $t \to 0$.

We show that the tail algebra of ψ_t is a subalgebra of $\mathcal{D}$.

Using results of [Voiculescu '99] on liberation Fisher information, it follows that $\mathcal{D} \cap \sigma_t(A)' = \mathbf{C}1$. Thus, the center of $\mathcal{M}_{\psi_t}$ is trivial and ψ_t is an extreme point of $\mathrm{TQSS}(A)$.