Poisson boundary of monoidal categories

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Summary: categorical Poisson boundary

Input

1. C*-tensor category C with:
   - irreducible decomposition
   - conjugates
2. 'symmetric' probability measure μ on the irreducible classes of C

Output

Poisson boundary $P(C;\mu)$, another C*-tensor category

- C*-tensor functor $\Pi\colon C \to P(C;\mu)$
- $(\exists\mu)$ Π is equivalence $\Leftrightarrow$ C is amenable

Summary: categorical Poisson boundary

Input

1. rigid semisimple C*-tensor category C
2. 'symmetric' probability measure μ on the irreducible classes of C

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Ref.

Poisson boundary of:

- fusion algebra $(\mathbb{Z}[\mathrm{Irr}\, C], d)$ (Hiai–Izumi) $\rightsquigarrow$ $\mathrm{Mor}_{P(C;\mu)}(1,1)$
- discrete quantum groups (Izumi) $\rightsquigarrow$ $P(\mathrm{Rep}\, G,\mu) \leftrightarrow H^\infty(\hat{G},\mu)$

Compact quantum groups

G: compact semisimple Lie group, maximal torus $T < G$, $0 < q < \infty$

- Φ_q: Knizhnik–Zamolodchikov associator
- G_q: compact quantum group, $\operatorname{Rep} G_q \simeq (\operatorname{Rep} G, \Phi_q)$
- $T < G_q$

μ: symmetric probability measure on $\operatorname{Irr} G \simeq \operatorname{Irr} G_q$

1. $P(\operatorname{Rep} G; \mu) = \operatorname{Rep} G$
2. $P(\operatorname{Rep} G_q; \mu) = \operatorname{Rep} T$

$\rightsquigarrow$ classification of the quantum groups G' such that

1. $\exists$ categorical equivalence $\operatorname{Rep} G' \simeq \operatorname{Rep} G_q$
2. representations have the same dimension: $\dim H_\pi = \dim H'_\pi$

Random walk on discrete groups

1. Γ: discrete group
2. μ: (generating, symmetric) probability measure on Γ

- Markov operator: P_μ on $\ell^\infty\Gamma$:

$$(P_\mu f)(g) = \sum_{h \in G} \mu(h) f(gh)$$

- *Harmonic functions* $H^\infty(\Gamma;\mu) = \{f \in \ell^\infty\Gamma \mid P_\mu f = f\}$
- Product structure on $H^\infty(\Gamma;\mu)$:

$$f_0 \cdot f_1 = \mathrm{w}^*\text{-}\lim_{n\to\infty} P_\mu^n(f_0 f_1)$$

- *Poisson boundary* $\partial_\mu\Gamma = \mathrm{Spec}(H^\infty(\Gamma;\mu))$

C*-tensor category

Definition (C*-tensor category)

Category C with:

1. Banach norm and involution on the morphism sets
   - the C*-identity $\|T^*T\| = \|T\|^2 = \|TT^*\|$ for $T \in C(X, Y)$
2. 'tensor product' functor $C \times C \to C$, $(X, Y) \mapsto X \otimes Y$
3. 'tensor unit' object $1_C \in C$
4. associativity isomorphisms $\phi_{X,Y,Z} \colon X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$

Semisimplicity

Objects in C admit irreducible decomposition $\Leftrightarrow$ $\dim C(X, Y) < \infty$

C*-tensor functor

Definition (C*-tensor functor)

C, C': C*-tensor categories

A C^**-tensor functor* $\mathcal{F}\colon C \to C'$ is a *pair* (F, η):

1. C*-functor $F\colon C \to C'$: $F(T^*) = F(T)^*$
2. natural isomorphism $\eta\colon F(X \otimes Y) \to F(X) \otimes F(Y)$

satisfying compatibility with tensor units & associativity isomorphisms.

Example (twists for quantum groups)

- $\Phi \in LG \otimes LG \otimes LG$ associator for $\operatorname{Rep} G$ (3-cocycle on $\hat{G}$)
- $\eta \in LG \otimes LG$ *twist* for Φ: $\eta_{12}(\hat{\Delta} \otimes \iota)(\eta)(\iota \otimes \hat{\Delta})(\eta^*)\eta_{23}^* = \Phi$

$\rightsquigarrow$ C*-tensor functor $(\mathrm{Id}, \eta)\colon\ \operatorname{Rep} G \to (\operatorname{Rep} G, \Phi)$

C*-tensor functors and actions

Example

Discrete group Γ, measurable Γ-space $\Gamma \curvearrowright X$

- C*-tensor category $C_{\Gamma \curvearrowright X}$
  1. objects Γ, morphisms $C_{\Gamma \curvearrowright X}(g, h) = L^\infty(X)\delta_{g,h}$
  2. monoidal structure:
     $f_0 \in C(g, g), f_1 \in (h, h) \Rightarrow f_0 \otimes f_1 = f_0(x) f_1(g^{-1}x) \in C(gh, gh)$
- C*-tensor functor $C_\Gamma \to C_{\Gamma \curvearrowright X}$
- "cocycle action" ⇔ general C*-tensor functor $C_\Gamma \to C$

Example

G: compact quantum group

Tensor functors $\operatorname{Rep} G \to C$

⇔ braided commutative Yetter–Drinfeld G-algebras

Conjugate objects

Definition (Conjugate objects)

Given C^*-tensor category C and $X \in \operatorname{Obj} C$, we say $Y \in \operatorname{Obj} C$ is a *conjugate* of X if $\exists R \in C(1_C, Y \otimes X), \bar{R} \in C(1_C, X \otimes Y)$ s.t.

$$(\bar{R}^* \otimes \iota)(\iota \otimes R) = \iota_X, \qquad (R^* \otimes \iota)(\iota \otimes \bar{R}) = \iota_Y.$$

$d(X) = \min_{(R,\bar{R})} \|R\| \|\bar{R}\|$ is called the *intrinsic dimension* of X

Objects in C have conjugates & $1_C \in C$ is irreducible $\rightsquigarrow$

- $d(X \otimes Y) = d(X)d(Y)$, $d(X \oplus Y) = d(X) + d(Y)$.
- any object have irreducible decomposition

Example

$C = \operatorname{Rep} G_q$: $d(X) = \operatorname{Tr}_X(q^\rho)$, quantum dimension of $X \in C$

Amenability of monoidal categories

C: semisimple ($\exists$ irreducible decomposition) C*-tensor category

- the Hilbert space $\ell^2 \operatorname{Irr} C$ from the irreducible classes $\operatorname{Irr} C$
- Γ_X: 'fusion' operator $\Gamma_X \delta_U = \sum_{V\colon \operatorname{Irr} C} \operatorname{mult}(V, X \times U) \delta_V$
- C has a dimension function $d \Rightarrow \|\Gamma_X\| \leq d(X)$

Definition

A rigid semisimple C*-tensor category C is *amenable* if $\|\Gamma_X\| = d(X)$ (intrinsic dimension) for any $X \in C$

Example

G: compact quantum group $\rightsquigarrow \operatorname{Rep} G$ is amenable $\Leftrightarrow$ G is of coamenable Kac type (no G_q!)

Dual category

C: C*-tensor category, $X \in C$
$\rightsquigarrow$ functor $\iota \otimes X \colon C \to C, Y \to Y \otimes X$

The 'dual' category $\hat{C}$:

1. Start from the objects of C, with enlarge morphism sets

$$\hat{C}(X, Y) = \mathrm{Nat}_b(\iota \otimes X, \iota \otimes Y) \simeq \ell^\infty_{U \colon \operatorname{Irr} C} C(U \otimes X, U \otimes Y)$$

   uniformly bounded natural transformations

2. Take subobject completion (projections in $\hat{C}(X, X)$ define subobjects of X)
3. Monoidal structure

$$\phi \otimes \iota_W = (\phi_U \otimes \iota_W)_U, \quad \iota_W \otimes \phi = (\phi_{U \otimes W})_U, \quad \phi \otimes \psi = (\phi \otimes \iota)(\iota \otimes \psi)$$

Categorical Poisson boundary

C: C*-tensor category with conjugates & irreducible unit
(⇒ irreducible decomposition of arbitrary objects)

1. Partial trace:

$$\mathrm{tr}_Z \otimes \iota \colon C(Z \otimes X, Z \otimes Y) \to C(X, Y),\ T \mapsto (R_Z^* \otimes \iota_Y)(\iota_{\bar{Z}} \otimes T)(R_Z \otimes \iota_X)$$

2. CP map on $\hat{C}(X, Y)$: $(\mathrm{tr}_Z \otimes \iota)(\phi) = (\mathrm{tr}_Z \otimes \iota)(\phi_{Z \otimes U})_U$

Definition

μ: probability measure on $\operatorname{Irr} C$

1. *Markov operator* on $\hat{C}(X, Y)$: $P_\mu = \sum_U \mu(U)(\mathrm{tr}_U \otimes \iota)$
2. *Harmonic family*: $(\eta_X)_X \in \mathrm{Nat}_b(\iota \otimes U, \iota \otimes V)$ s.t. $P_\mu \eta = \eta$

Categorical Poisson boundary, cont.

Categorical Poisson boundary $P(C, \mu)$:

- Start from the objects in C
- $\mathrm{Mor}_{P(C,\mu)}(U, V) = \{\text{harmonic family } (\eta_X)_X \in \mathrm{Nat}_b(\iota \otimes U, \iota \otimes V)\}$
- Composition in $P(C, \mu)$: $(\xi \cdot \eta)_X = \lim_{n \to \infty} P_\mu^n(\xi\eta)_X$
- Take the subobject completion
- 'Same' monoidal structure as $\hat{C}$

Tensor functor $\Pi\colon C \to P(C; \mu)$, $X \mapsto (\iota_U \otimes X)_U$ "constant family"

Example

1. Γ discrete group $\Rightarrow P(C_\Gamma; \mu) = C_{\Gamma \curvearrowright \partial_\mu \Gamma}$
2. G compact quantum group
   $\Rightarrow$ $(\mathrm{Rep}\, G \to P(\mathrm{Rep}\, G; \mu)) \leftrightarrow$ YD-G-algebra $H^\infty(\hat{G}; \mu)$

Amenability and Poisson boundary

Theorem (Neshveyev–Y.)

C: C^*-tensor category with conjugates and irreducible unit

C is amenable $\Leftrightarrow$ $\exists$ prob. measure μ on $\operatorname{Irr} C$ such that $\Pi\colon C \to P(C;\mu)$ is an equivalence of C^*-tensor categories.

Key construction: with $V = \operatorname{supp} \mu$,

1. $N_X^{(n)} = C(V^{\otimes n} \otimes X, V^{\otimes n} \otimes X)$ has a state $\omega^{(n)} = \operatorname{tr}_X P_\mu^n$.
2. Put $N_X = \lim_n (N_X^{(n)}, \omega^{(n)})$ as a von Neumann algebra.
3. We have $\operatorname{Mor}_{P(C;\mu)}(X, X) = N_X \cap N_1'$. (cf. Longo–Roberts, Hayashi–Yamagami)

$\Rightarrow$: Use the universality of $(R, \bar{R})$ in C $\Rightarrow$ $C(X, X) \subset \operatorname{Mor}_{P(C;\mu)}(X, X)$ is 'dense'.

Amenability and Poisson boundary, cont.

Theorem (Neshveyev–Y.)

C: C^*-tensor category with conjugates and irreducible unit

C is amenable if and only if there is a probability measure μ on $\operatorname{Irr} C$ such that $\Pi\colon C \to P(C;\mu)$ is equivalence of C^*-tensor categories.

$\Leftarrow$: In general we have

$$\lim H(N_X^{(n)}|N_1^{(n)}) \leq 2\log\|\Gamma_X\| \leq -\lim\log\lambda(N_X^{(n)}, N_1).$$

From the assumption we get extremality of $N_1 \subset N_X$ from $C = P(C;\mu)$. Looking at the matricial inclusions $N_1^{(n)} \subset N_X^{(n)}$, we get the equality

$$\lim H(N_X^{(n)}|N_1^{(n)}) = H(N_X|N_1) = -\log\lambda(N_X, N_1)$$
$$= -\lim\log\lambda(N_X^{(n)}, N_1^{(n)}) = 2\log d(X).$$